Numerical Integration

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slides based on lecture notes/slides from L. Dede/S. Deparis

October 16/30, 2024

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Simple and *Composite* integration formulas

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Motivation

The integration is one of the biggest problem that we need to solve in analysis. Indeed, we often use integrals whose calculation by analytical method is difficult or even impossible, because there do not exists analytical expression of the integral. Here are some examples:

$$\int_0^1 e^{-x^2}\,dx, \qquad \int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx, \qquad \int_0^1 \cos x^2\,dx.$$

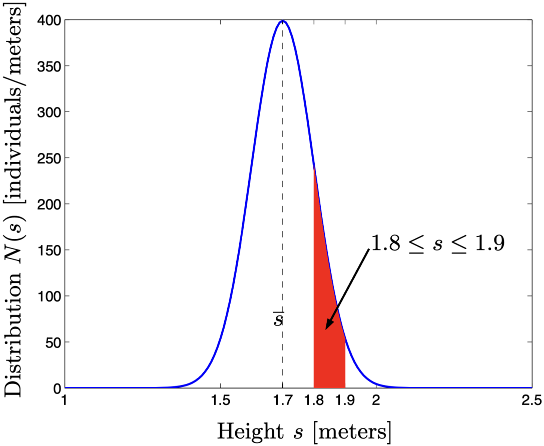
In these cases, we can apply numerical methods to estimate the value of the given integral.

Example

Example 1. We consider a population of a very large number M of individuals and we have the height of each individual. The distribution $N(s)$ of their height (such that ΔN represents the number of individuals whose height is between s and $s + \Delta s$ (written also $N(s)\Delta s$)) can be represented by a "bell" function characterized by the mean value $\overline{s}$ of the height and the standard deviation σ:

$$N(s) = \frac{M}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(s-\overline{s})^2}{2\sigma^2}\right).$$

Example (contd)



An instance is provided in the above figure ($M = 100$ individuals, $\bar{h} = 1.7$ meters, $\sigma = 0.1$ meters). The area of the red region gives the number of individuals whose height is between 1.8 and 1.9 meters.

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Simple integration formulas

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on an interval $[a, b] \subset \mathbb{R}$. We need to calculate the following quantity by numerical methods

$$I(f) = \int_a^b f(x)dx.$$

- We approximate $f(x)$ in $[a, b]$ with $\tilde{f}(x)$ which is easy to integrate in $[a, b]$, and obtain
$$I_q(f) = I(\tilde{f}) = \int_a^b \tilde{f}(x)dx.$$
- Typically for simple formulas, $\tilde{f}(x)$ is a polynomial of degree n interpolating $f(x)$ in $n + 1$ nodes in $[a, b]$

Composite integration formulas

We consider M subintervals $I_k = [x_{k-1}, x_k]$, $k = 1, \dots, M$, where $x_k = a + kH$ and $H = (b - a)/M$. Then we have

$$I(f) = \sum_{k=1}^{M} \int_{I_k} f(x)dx,$$

We can approximate the exact integral of f on each subinterval I_k by the integral of a polynomial approximating f on I_k, i.e.:

$$I(f) \text{ approximated by } \sum_{k=1}^{M} \int_{I_k} \Pi_n f(x)dx = \int_a^b \Pi_n^H f(x)dx.$$

Degree of exactness

Definition

A quadrature rule $I_q(f)$ on the interval $[a, b]$ is exact for a function f if

$$I_q(f) = \int_a^b f(x)dx.$$

It is **exact of degree r** if it is exact for all polynomials of degree less or equal to r, i.e.

$$I_q(p) = \int_a^b p(x)dx \quad \forall p \in \mathbb{P}_r,$$

but not for all those of degree $r + 1$. r is called a *degree of exactness* of the quadrature rule. (In other words, the degree of exactness of the quadrature rule is the largest integer $r \geq 0$ for which the approximate value of the integral of any polynomial of degree less or equal to r (obtained by the quadrature rule) is equal to the exact value).

Simple and *composite* integration formulas

We consider the following integration formula (called *simple*):

- Mid-point formula
- Trapezoidal formula
- Simpson formula

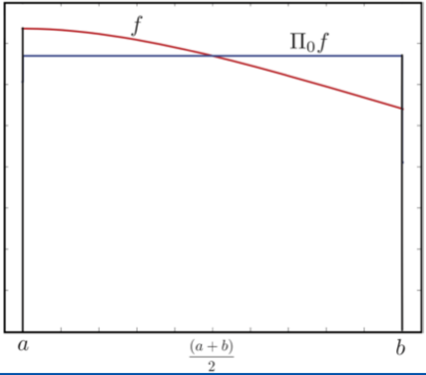
and their composite formulations.

Mid-point formula

Consider $f \in C^0([a,b])$,

$$I_{mp}(f) = I(\Pi_0 f) = (b-a) f\left(\frac{a+b}{2}\right), \qquad (1)$$

where $\Pi_0 f(x)$ is the polynomial of degree 0 interpolating $f(x)$ at the midpoing $\bar{x} = \frac{a+b}{2}$.



Composite mid-point quadrature formula

This formula is obtained by replacing, on each subinterval I_k, the function f with a constant polynomial $\Pi_0 f$ that is equal to the value of f in the middle of I_k (look at following figure) : We obtain the *composite midpoint quadrature formula*.

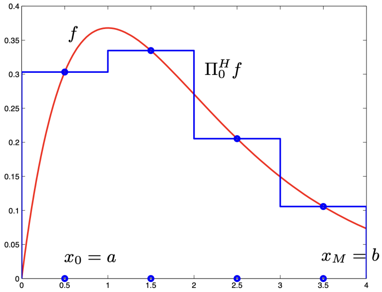
$$I^c_{mp}(f) := I(\Pi^H_0 f) = H \sum_{k=1}^{M} f(\overline{x}_k), \quad (2)$$

where

$$\overline{x}_k = \frac{x_{k-1} + x_k}{2},$$

and $\Pi^H_0 f(x)$ is the piecewise polynomial of degree 0 interpolating $f(x)$ at the mid-points $\{\bar{x}_k\}^M_{k=1}$

Composite mid-point quadrature formula



Integration error: mid-point quadrature

- Mid-point quadrature formula. If $f \in C^2([a,b])$, then

$$e_{mp}(f) := I(f) - I_{mp}(f) = \frac{(b-a)^3}{24} f''(\xi), \quad \text{for some } \xi \in [a,b]$$

- Composite mid-point quadrature formula. If $f \in C^2([a,b])$, then

$$e_{mp}^c(f) := I(f) - I_{mp}^c(f) = \frac{b-a}{24} H^2 f''(\xi), \quad \text{for some } \xi \in [a,b]$$

Definition

We define the **order** of an integration formula by the order of the error with respect to H.

Proof of error for simple midpoint formula

We will use the second mean value theorem of integration:
If $f, g \in C^0([a, b])$ and $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in ([a, b])$, then

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx \quad \text{for some } \xi \in [a, b]$$

Proof for simple midpoint formula

Consider Taylor expansion of $f(x)$ around $\bar{x}$, with $\bar{x} = \frac{a+b}{2}$,

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2} f''(\eta(x))(x - \bar{x})^2$$

for some $\eta(x) \in [a, b]$. Compute the integral of the expansion:

$$I(f) = I_{\mathrm{mp}}(f) + f'(\bar{x}) \int_a^b (x - \bar{x})\, dx + \frac{1}{2} f''(\xi) \int_a^b (x - \bar{x})^2\, dx,$$

for some $\xi \in [a, b]$, using the second mean value theorem. Since

$$\int_a^b (x - \bar{x})\, dx = 0 \;\text{ and }\; \int_a^b (x - \bar{x})^2\, dx = \frac{(b-a)^3}{12}, \quad \text{then}$$

$$e_{mp}(f) := I(f) - I_{mp}(f) = \frac{(b-a)^3}{24} f''(\xi), \;\; \text{for some } \xi \in [a, b]$$

Proof of error for composite midpoint formula

Use Taylor expansion on the interval $I_k = [x_{k-1}, x_k]$ in the point $\bar{x}_k = (x_{k-1} + x_k)/2$. We have

$$\int_{I_k} [f(x) - f(\bar{x}_k)]\, dx = \int_{I_k} f'(\bar{x}_k)(x - \bar{x}_k)dx + \frac{1}{2}\int_{I_k} f''(\xi(x))(x - \bar{x}_k)^2 dx,$$

where $\xi(x) \in I_k$. Furthermore, we have

$$\int_{I_k} f'(\bar{x}_k)(x - \bar{x}_k)dx = 0,$$

and thanks to mean value theorem for integration we have $\exists \xi_k \in I_k$:

$$\int_{I_k} f''(\xi(x))(x - \bar{x}_k)^2 dx = f''(\xi_k)\int_{I_k} (x - \bar{x}_k)^2 dx = \frac{H^3}{12} f''(\xi_k).$$

So:

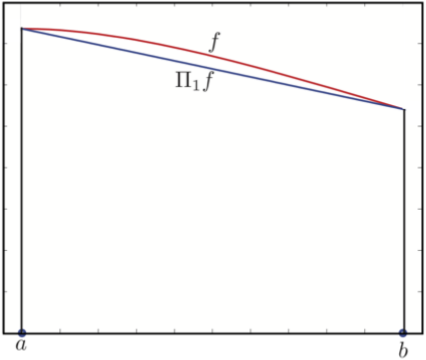
$$\int_{I_k} [f(x) - f(\bar{x}_k)]\, dx = \frac{H^3}{24} f''(\xi_k).$$

Trapezoidal formula

Consider $f \in C^0([a,b])$,

$$I_t(f) := I(\Pi_1 f) = (b-a)\frac{f(a)+f(b)}{2} \quad (3)$$

where $\Pi_1 f(x)$ is the polynomial of degree 1 interpolating $f(x)$ at the nodes a and b.



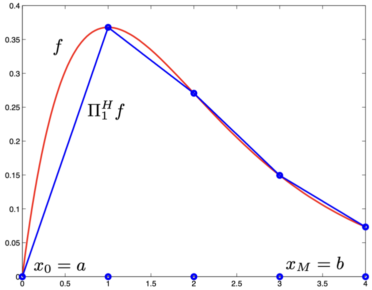
Composite trapezoidal formula

If, on each subinterval I_k, we replace f with the interpolating polynomial $\Pi_1 f(x)$ of degree 1 at the nodes x_{k-1} and x_k, then we obtain the *composite trapezoidal formula*:

$$I_t^c(f) := I(\Pi_1^H f) = \frac{H}{2}\sum_{k=1}^{M}[f(x_k) + f(x_{k-1})] = \frac{H}{2}[f(a) + f(b)] + H\sum_{k=1}^{M-1} f(x_k). \tag{4}$$

where $\Pi_1^H f(x)$ is the piecewise polynomial of degree 1 interpolating $f(x)$ and the nodes $\{x_k\}_{k=0}^{M}$ of the interval $[a, b]$.

Composite trapezoidal formula



Integration error: trapezoidal formula

- Trapezoidal formula. If $f \in C^2([a,b])$, then

$$e_t(f) := I(f) - I_t(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad \text{for some } \xi \in [a,b]$$

- Composite trapezoidal formula. If $f \in C^2([a,b])$, then

$$e_t^c(f) := I(f) - I_t^c(f) = -\frac{b-a}{12} H^2 f''(\xi), \quad \text{for some } \xi \in [a,b]$$

Degree of exactness for mid-point quadrature

Remark The mid-point quadrature formulas have degree of exactness 1

- If $f \in P_1$, $f''(x) = 0$ for all $x \in \mathbb{R}$
- The errors $e_{\mathrm{mp}}(f)$ and $e_{\mathrm{cmp}}(f)$ are identically zero for all polynomials of degree less than or equal to 1.

Integration error: simple formulas

- Mid-point quadrature formula. If $f \in C^2([a,b])$, then

$$e_{mp}(f) := I(f) - I_{mp}(f) = \frac{(b-a)^3}{24} f''(\xi), \quad \text{for some } \xi \in [a,b]$$

- Trapezoidal formula. If $f \in C^2([a,b])$, then

$$e_t(f) := I(f) - I_t(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad \text{for some } \xi \in [a,b]$$

Integration error: composite formulas

- Composite mid-point quadrature formula. If $f \in C^2([a,b])$, then

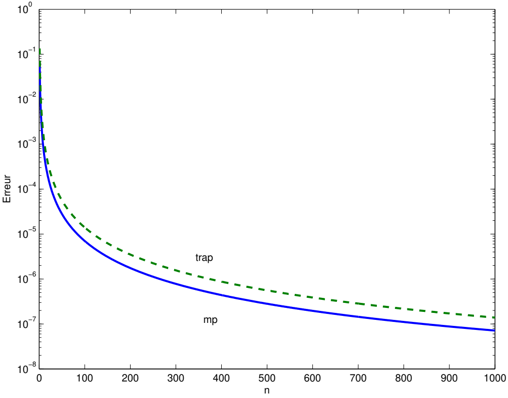
$$e_{mp}^c(f) := I(f) - I_{mp}^c(f) = \frac{b-a}{24} H^2 f''(\xi), \quad \text{for some } \xi \in [a,b]$$

- Composite trapezoidal formula. If $f \in C^2([a,b])$, then

$$e_t^c(f) := I(f) - I_t^c(f) = -\frac{b-a}{12} H^2 f''(\xi), \quad \text{for some } \xi \in [a,b]$$

Example

We consider $I(f) = \int_0^1 f(x)dx$ where $f(x) = \cos(x^2)$: the following figure shows the error of integration $|I^c_{mp}(f) - I(f)|$ (composite midpoint quadrature formula) and $|I^c_t(f) - I(f)|$ (composite trapezoidal formula) with respect to the number of subintervals M.

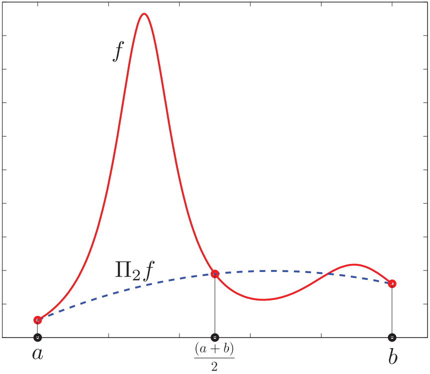


Simpson formula

Consider $f \in C^0([a,b])$,

$$I_s(f) := I(\Pi_2 f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad (5)$$

where $\Pi_2 f(x)$ is the polynomial of degree 2 interpolating $f(x)$ at the nodes a, b, and $\frac{a+b}{2}$.



Composite Simpson quadrature formula

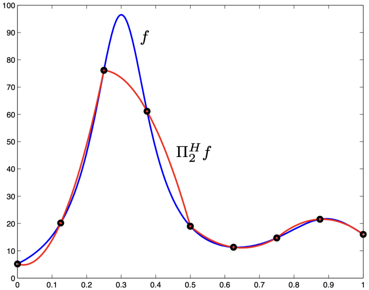
The Simpson formula can be obtained by replacing f by composite interpolating polynomial $\Pi_2^H f(x)$ of degree 2. In particular, $\Pi_2^H f(x)$ is a composite continuous function which on each subinterval I_k is obtained as the interpolating polynomial of f with nodes

$$x_{k-1},\ \overline{x}_k = \frac{x_{k-1} + x_k}{2} \text{ and } x_k \text{ (see the following figure).}$$

Then we obtain the *composite Simpson quadrature formula*:

$$I_s^c(f) := I(\Pi_2^H f) = \frac{H}{6} \sum_{k=1}^{M} \left[f(x_{k-1}) + 4f(\overline{x}_k) + f(x_k) \right]. \tag{6}$$

Composite Simpson quadrature formula



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Integration error: simple formulas

- Mid-point quadrature formula. If $f \in C^2([a,b])$, then

$$e_{mp}(f) := I(f) - I_{mp}(f) = \frac{(b-a)^3}{24} f''(\xi), \text{ for some } \xi \in [a,b]$$

- Trapezoidal formula. If $f \in C^2([a,b])$, then

$$e_t(f) := I(f) - I_t(f) = -\frac{(b-a)^3}{12} f''(\xi), \text{ for some } \xi \in [a,b]$$

- Simpson quadrature formula. If $f \in C^4([a,b])$, then

$$e_s(f) := I(f) - I_s(f) = -\frac{(b-a)^5}{180 \cdot 16} f^{(4)}(\xi), \text{ for some } \xi \in [a,b]$$

Integration error: composite formulas

- Composite mid-point quadrature formula. If $f \in C^2([a,b])$, then

$$e^c_{mp}(f) := I(f) - I^c_{mp}(f) = \frac{b-a}{24} H^2 f''(\xi), \text{ for some } \xi \in [a,b]$$

- Composite trapezoidal formula. If $f \in C^2([a,b])$, then

$$e^c_t(f) := I(f) - I^c_t(f) = -\frac{b-a}{12} H^2 f''(\xi), \text{ for some } \xi \in [a,b]$$

- Composite Simpson quadrature formula. If $f \in C^4([a,b])$, then

$$e^c_s(f) := I(f) - I^c_s(f) = -\frac{b-a}{180 \cdot 16} H^4 f^{(4)}(\xi), \text{ for some } \xi \in [a,b]$$

Definition

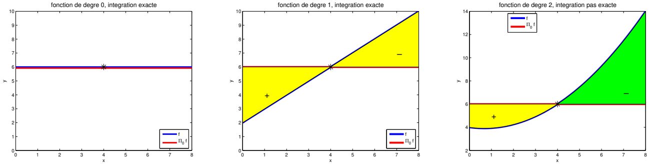
We define the **order** of an integration formula by the order of the error with respect to H.

Degree of exactness

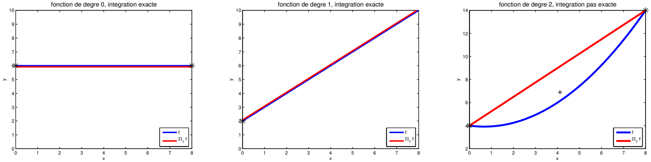
Let us take into account the simple midpoint formula, the trapezoidal and the Simpson. We can link a degree of exactness to the formulas.
In particular, we can show that I_{pm} and I_t has degree of exactness equal to 1; the Simpson formula has degree of exactness equal to 3.

Composite formula	*Dg. of exact.*	*Ord. with respect to H*
Midpoint (2)	1	2
Trapezoidal (4)	1	2
Simpson (6)	3	4

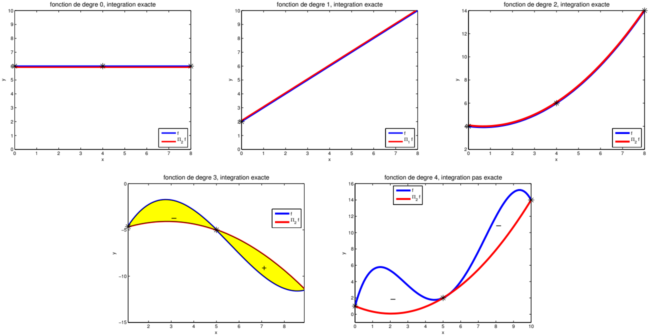
Midpoint formula



Trapezoidal formula



Simpson formula

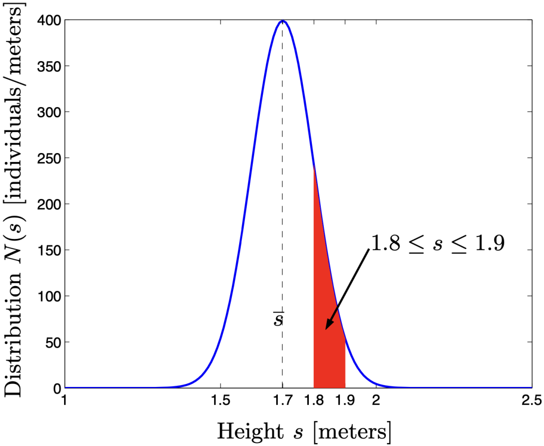


Example

Example 1. We consider a population of a very large number M of individuals and we have the height of each individual. The distribution $N(s)$ of their height (such that ΔN represents the number of individuals whose height is between s and $s + \Delta s$ (written also $N(s)\Delta s$)) can be represented by a "bell" function characterized by the mean value $\bar{s}$ of the height and the standard deviation σ:

$$N(s) = \frac{M}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(s-\bar{s})^2}{2\sigma^2}\right).$$

Example (contd)



An instance is provided in the above figure ($M = 100$ individuals, $\bar{h} = 1.7$ meters, $\sigma = 0.1$ meters). The area of the red region gives the number of individuals whose height is between 1.8 and 1.9 meters.

Example 1 (contd). Let us consider the example of computing the height of individuals. To compute the number of individuals whose height is between 1.8 and 1.9 meters we use the composite Simpson formula with 100 subintervals (`simpsonc` command):

```
>>N = @(h,M,hbar,sigma) M/(sigma*sqrt(2*pi))*exp(-(h-hbar).^2./(2*
>> M = 100; hbar = 1.7; sigma = 0.1;
>> int = simpsonc(1.8, 1.9, 100, N, M, hbar, sigma)
ans =
   13.5905
```

We therefore estimate that the number of individuals in this range of height is 13.6 % of the population.

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Interpolatory Quadrature Formulas

- Gauss–Legendre quadrature formulas
- Gauss–Legendre-Lobatto quadrature formulas

Numerical Integration in Multiple Dimensions

Interpolatory quadrature formulas

Goal: Provide a generalization of the previous simple formulas.

Definition (Definition 5.6)

Let us consider a function $f(x) \in C^0([a, b])$. Then a (simple) interpolatory quadrature formula is defined as:

$$\tilde{I}(f) := I(\tilde{f}) = \sum_{j=0}^{n} \alpha_j f(y_j),$$

where $\tilde{f}(x)$ is a function interpolating $f(x)$ at the $n+1$ quadrature nodes $\{y_j\}_{j=0}^n \subset [a, b]$, and $\{\alpha_j\}_{j=0}^n$ are the corresponding quadrature weights, with $n \geq 0$.

Different choices possible for $\tilde{f}(x)$, which should be easy to integrate.

Interpolatory quadrature formulas

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Different choices possible for $\tilde{f}(x)$, which should be easy to integrate.

Using Lagrange interpolating polynomials

In general, if we choose $\tilde{f}(x) = \Pi_n f(x)$

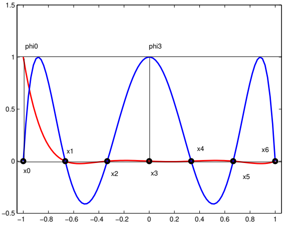
$$I(\tilde{f}) = \int_a^b \Pi_n f(x) dx,$$

where $\Pi_n f$ is the Lagrange interpolating polynomial (of degree $n \geq 0$) of the function f at the nodes $x_0, \ldots, x_n$:

$$I(\tilde{f}) = \int_a^b \Pi_n f(x) dx = \int_a^b \sum_{k=0}^{n} f(x_k)\varphi_k(x) dx = \sum_{k=0}^{n} \underbrace{\left[\int_a^b \varphi_k(x) dx\right]}_{\alpha_k} f(x_k)$$

$$\varphi_k \in \mathbb{P}_n : \varphi_k(x_i) = \delta_{ik}, \quad k, i = 0, \cdots, n$$

is the k-th characteristic Lagrange polynomial.



Using Lagrange interpolating polynomials

We have the general formula:

$$\tilde{I}(f) := I(\tilde{f}) = \sum_{j=0}^{n} \alpha_j f(x_j), \tag{7}$$

where x_j are the quadrature nodes and α_j are the quadrature weights (look at the following table).

Formula	x_k	α_k
Midpoint (1)	$x_0 = \frac{1}{2}(a+b)$	$\alpha_0 = b - a$
Trapezoidal (3)	$x_0 = a,\ x_1 = b$	$\alpha_0 = \alpha_1 = \frac{1}{2}(b-a)$
Simpson (5)	$x_0 = a,\ x_1 = \frac{1}{2}(a+b),$	$\alpha_0 = \alpha_2 = \frac{1}{6}(b-a),$
	$x_2 = b$	$\alpha_1 = \frac{2}{3}(b-a)$

Error with Lagrange interpolating polynomials

The integration error is given by:

$$\begin{aligned}|I(f) - I(\tilde{f})| &= \left| \int_a^b f(x)dx - \int_a^b \Pi_n f(x)dx \right| \\ &= \left| \int_a^b (f - \Pi_n f)(x)dx \right| \\ &\leq \underbrace{\max_{x \in [a,b]} |f(x) - \Pi_n f(x)|}_{\text{interpolation error}} (b - a)\end{aligned}$$

Increasing n is not a good strategy to reduce the integration error $|I(f) - I(\tilde{f})|$.

Integrating constant functions

Minimum objective: exactly integrate constant functions $f(x) = C$ for any $n \geq 0$.
Since $I(f) = I(C) = C(b-a)$, we set

$$\sum_{j=0}^{n} \alpha_j f(y_j) = \sum_{j=0}^{n} \alpha_j C = C(b-a),$$

for which we obtain the following condition on the quadrature weights:

$$\sum_{j=0}^{n} \alpha_j = b - a \quad \text{for all } n \geq 0,$$

regardless of the position of the quadrature nodes.

Reference interval

Provide general quadrature formulas that can be applied to functions $f(x)$ in any interval $[a, b]$ by using a reference interval:

- Specify quadrature nodes $\{\bar{y}_j\}_{j=0}^{n}$ and weights $\{\bar{\alpha}_j\}_{j=0}^{n}$ in the reference interval $[-1, 1]$
- Recover quadrature nodes and weights for the general interval $[a, b]$ as:

$$y_j = \frac{a+b}{2} + \frac{b-a}{2}\bar{y}_j \quad \text{for } j = 0, \ldots, n,$$

and

$$\alpha_j = \frac{b-a}{2}\bar{\alpha}_j \quad \text{for } j = 0, \ldots, n,$$

respectively.

Gauss–Legendre quadrature formulas

Proposition

For $m > 0$, the quadrature formula $\sum_{j=0}^{n} \bar{\alpha}_j f(\bar{y}_j)$ has degree of exactness $m+n$ iff it is of interpolatory type and $\omega_{n+1}(x) = \prod_{i=0}^{n}(x - \bar{y}_i)$ satisfies:

$$\int_{-1}^{1} \omega_{n+1}(x) p(x) dx = 0, \quad \text{for all } p \in \mathbb{P}_{m-1}$$

Corollary

The maximum degree of exactness is $r = 2n+1$.
This is given by taking $\omega_{n+1}(x)$ proportional to the Legendre polynomial $L_{n+1}(x)$ of degree $n+1$.

Gauss–Legendre quadrature formulas

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Corollary

The maximum degree of exactness is $r = 2n + 1$.
This is given by taking $\omega_{n+1}(x)$ proportional to the Legendre polynomial $L_{n+1}(x)$ of degree $n + 1$.

Gauss–Legendre quadrature formulas

- Family of interpolatory quadrature formulas that approximate $f(x)$ using Legendre polynomials
- Legendre polynomials $\{L_k(x)\}_{k=0}^{n+1}$ in the interval $[-1,1]$ are recursively defined as:

$$
\begin{aligned}
L_0(x) &= 1, \\
L_1(x) &= x, \\
L_{k+1}(x) &= \frac{2k+1}{k+1} x L_k(x) - \frac{k}{k+1} L_{k-1}(x) \quad \text{for } k = 1, \ldots, n.
\end{aligned}
$$

- Legendre polynomials are orthogonal:

$$\int_{-1}^{1} L_{n+1}(x) L_k(x)\, dx = 0 \quad \text{for all } k = 0, \ldots, n.$$

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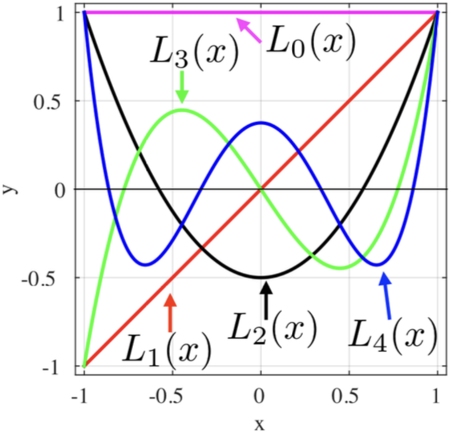
Exemple of Legendre polynomials

Consider Legendre polynomials for $n = 3$ in $[-1, 1]$:

$$
\begin{aligned}
L_0(x) &= 1, \\
L_1(x) &= x, \\
L_2(x) &= \frac{3}{2}xL_1(x) - \frac{1}{2}L_0(x), \\
L_3(x) &= \frac{5}{3}xL_2(x) - \frac{2}{3}L_1(x), \\
L_4(x) &= \frac{7}{4}xL_3(x) - \frac{3}{4}L_2(x).
\end{aligned}
$$



Gauss–Legendre quadrature formulas

Definition (5.7)

Let us consider a function $f(x) \in C^0([a, b])$. Then the Gauss–Legendre quadrature formula for $n \geq 0$ over the reference interval $[-1, 1]$ is:

$$I_{\mathrm{GL},n} = \sum_{j=0}^{n} \bar{\alpha}_j^{\mathrm{GL}} f\left(\bar{y}_j^{\mathrm{GL}}\right),$$

where:

$$\bar{y}_j^{\mathrm{GL}} := \text{zeros of } L_{n+1}(x) \text{ for all } j = 0, \ldots, n,$$

$$\bar{\alpha}_j^{\mathrm{GL}} := \frac{2}{\left[1 - \left(\bar{y}_j^{\mathrm{GL}}\right)^2\right] \left[L'_{n+1}\left(\bar{y}_j^{\mathrm{GL}}\right)\right]^2} \text{ for all } j = 0, \ldots, n.$$

Formulas and degree of exactness

- Degree of exactness of the Gauss–Legendre quadrature formula is $r = 2n + 1$ for all $n \geq 0$.
- Quadrature nodes and weights of the Gauss–Legendre quadrature formulas over $[-1, 1]$ for $n = 0, 1, 2$:

n	Nodes $\{\bar{y}_j^{\mathrm{GL}}\}_{j=0}^n$	Weights $\{\bar{\alpha}_j^{\mathrm{GL}}\}_{j=0}^n$	r
0	0	2	1 (mid-point formula)
1	$\left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$	$\{1, 1\}$	3
2	$\left\{-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}\right\}$	$\left\{\frac{5}{9}, \frac{8}{9}, \frac{5}{9}\right\}$	5

Gauss–Legendre-Lobatto quadrature

Extend the concept of maximizing the degree of exactness by including the boundaries of the interval as quadrature nodes.

Definition (5.8)

Let us consider a function $f(x) \in C^0([a,b])$. Then the Gauss–Legendre–Lobatto quadrature formula for $n \geq 1$ over the reference interval $[-1,1]$ is:

$$I_{\mathrm{GLL},n} = \sum_{j=0}^{n} \bar{\alpha}_j^{\mathrm{GLL}} f\left(\bar{y}_j^{\mathrm{GLL}}\right),$$

where:

$$\bar{y}_0^{\mathrm{GLL}} := -1, \quad \bar{y}_n^{\mathrm{GLL}} := +1, \quad \text{and} \quad \bar{y}_j^{\mathrm{GLL}} := \text{zeros of } L_n'(x) \text{ for all } j = 1, \ldots, n-1.$$

$$\bar{\alpha}_j^{\mathrm{GLL}} := \frac{2}{n(n+1)} \frac{1}{\left(L_n\left(\bar{y}_j^{\mathrm{GLL}}\right)\right)^2} \quad \text{for all } j = 0, \ldots, n.$$

Formulas and degree of exactness

- Degree of exactness of Gauss–Legendre-Lobatto quadrature is $r = 2n - 1$ for all $n \geq 1$.
- Quadrature nodes and weights of Gauss–Legendre-Lobatto quadrature over $[-1, 1]$ for $n = 1, 2, 3$:

n	Nodes $\{\bar{y}_j^{\text{GLL}}\}_{j=0}^{n}$	Weights $\{\bar{\alpha}_j^{\text{GLL}}\}_{j=0}^{n}$	r
1	$\{-1, +1\}$	$\{1, 1\}$	1 (trapezoidal formula)
2	$\{-1, 0, +1\}$	$\left\{\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right\}$	3 (Simpson's formula)
3	$\left\{-1, -\frac{1}{\sqrt{5}}, +\frac{1}{\sqrt{5}}, +1\right\}$	$\left\{\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{6}\right\}$	5

Plan

Numerical Integration in Multiple Dimensions

- Integration of continuous functions $f : \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^d$ for $d \geq 2$, based on generalization of quadrature formulas
- The formula will be:
$$I(f) = \int_{\Omega} f(x)dx$$
- Simple formulas defined in reference domains, as e.g. trapezoids and triangles for $d = 2$ or tetrahedrons for $d = 3$
- Composite formulas used for complex domains

Example of numerical quadrature in 2D