EPFL

Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Solutions – Approximation of functions and data

Solution II (MATLAB)

a) We execute the following commands:

```
f = @(x) sin( x );   a = 0;   b = 3 * pi;
n_vect = 1 : 7;   % vector containing all the degrees of desired polynomials
x_values = linspace( a, b, 1001 );
f_values = f( x_values );
for n = n_vect    % for all the degrees in n_vect
    x_nodes = linspace( a, b, n + 1 );
    y_nodes = f( x_nodes );
    P = polyfit( x_nodes, y_nodes, n );
    P_values = polyval( P, x_values );
    figure( n );
    plot( x_values, P_values, '-k', ...
              x_values, f_values, '--k', x_nodes, y_nodes, 'xk'  );
    legend( '\Pi_n f(x)', 'f(x)', '(x_i,y_i)');
end
```

We obtain the results reported in Figure 1 $n = 2, 3, 5$, and 6. We observe the convergence of the interpolating polynomials $\Pi_n f(x)$ to $f(x)$ for increasing values of n. For $n = 3$, we observe that the data points are aligned on a horizontal line, so that $\Pi_3 f(x) = c \in \mathbb{R}$; more specifically, we obtain that $\Pi_3 f(x) = 0$.

b) We compute the error as follows:

```
f = @(x) sin( x );   a = 0;   b = 3 * pi;
n_vect = 1 : 7;   % vector containing all the degrees of desired polynomials
x_values = linspace( a, b, 1001 );
f_values = f( x_values );
err = [ ];   % initialization of the vector containing the true errors
for n = n_vect
    x_nodes = linspace( a, b, n + 1 );
```

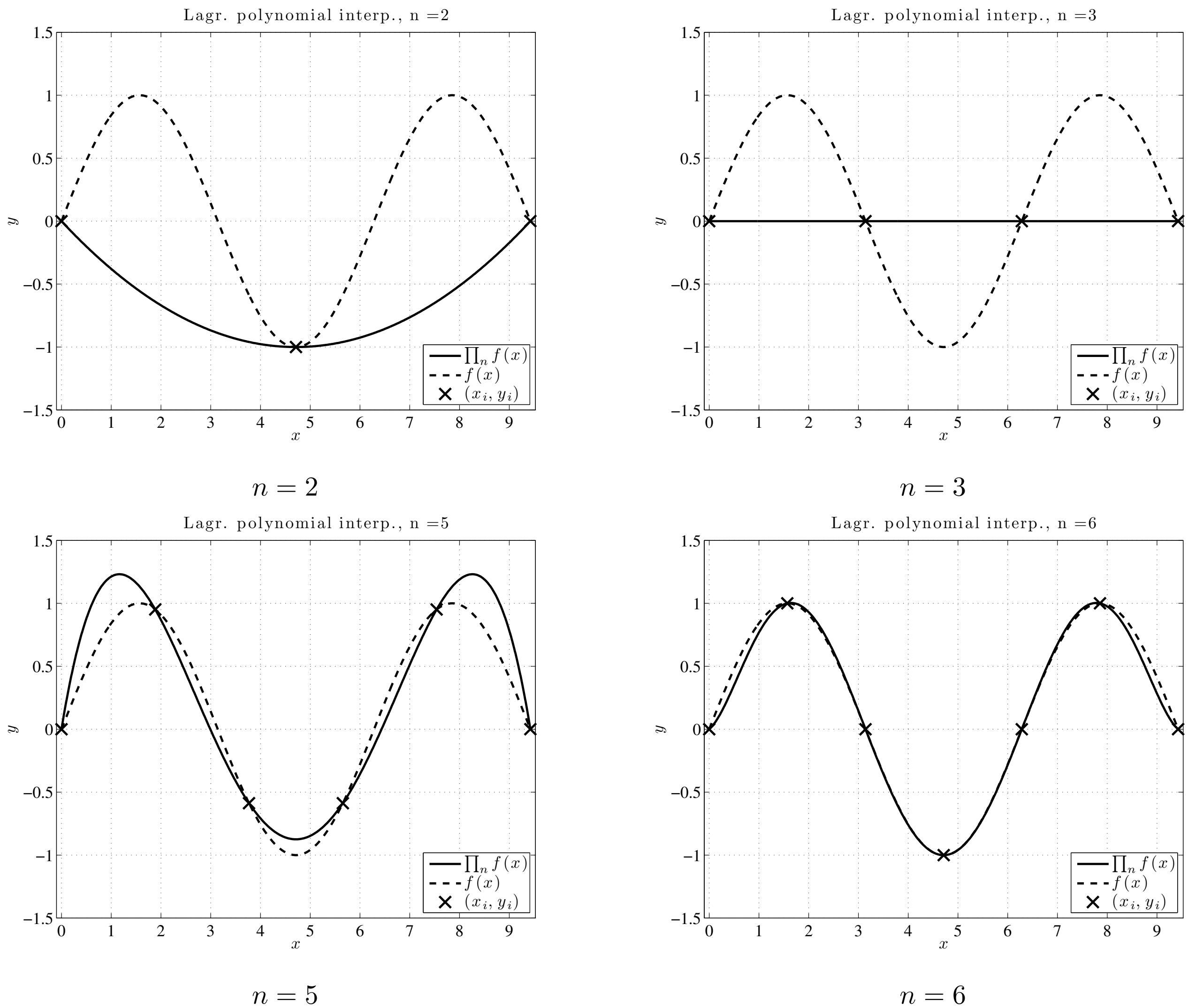

Figure 1: Interpolating polynomials $\Pi_n f(x)$ of the function $f(x) = \sin(x)$ at uniformly spaced nodes in $I = [0, 3\pi]$ for $n = 2, 3, 5$, and 6.

```
    y_nodes = f( x_nodes );
    P = polyfit( x_nodes, y_nodes, n );
    P_values = polyval( P, x_values );
    err = [ err, max( abs( P_values - f_values ) ) ];  % append errors to err
end
err
%   err =
%       1.0000     1.5925     1.0000     0.6363     0.4228     0.1301     0.0895
plot( n_vect, err, '-ko' );
```

As we can observe from Figure 2 (left), the error $e_n(f)$ decreases when n increases.

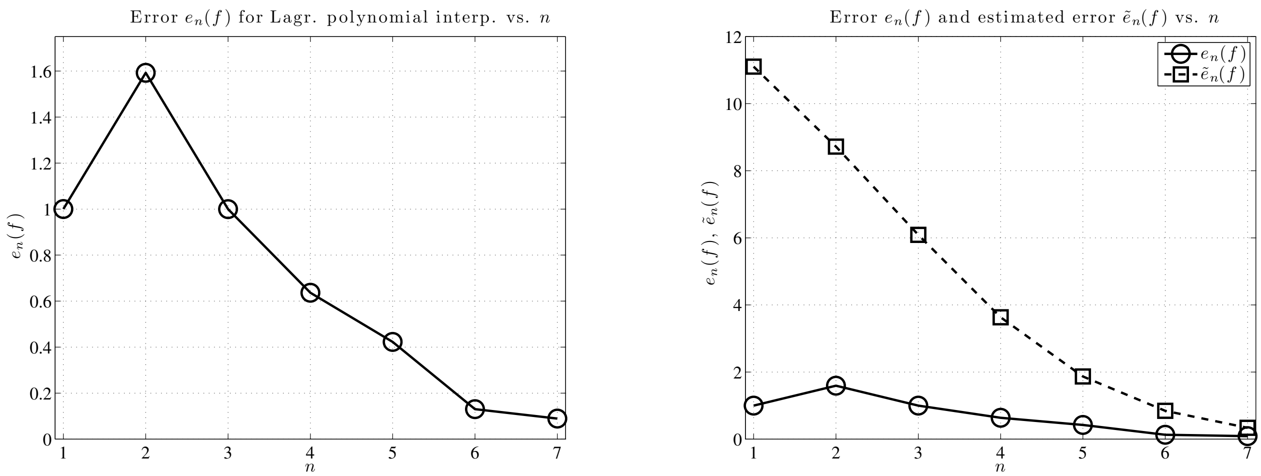

Figure 2: Errors $e_n(f)$ vs. n for the interpolating polynomials $\Pi_n f(x)$ of the function $f(x) = \sin(x)$ (left) and comparison with the error estimator $\widetilde{e}_n(f)$ (right).

c) We observe that $\max_{x \in I} \left| f^{(n+1)}(x) \right| = 1$, since $f^{(1)}(x) = \cos(x)$, $f^{(2)}(x) = -\sin(x)$, $f^{(3)}(x) = -\cos(x)$, ... As a consequence, the error estimator reads $\widetilde{e}_n(f) = \frac{1}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1}$, which is monotonically decreasing when n increases. We plot in Figure 2 (right) the error estimator $\widetilde{e}_n(f)$ by means of the following commands:

```
err_estimated = [ ];
for n = n_vect
       df_max = 1;  % for all n and x \in I=[0,3 *pi]
       err_estimated = [ err_estimated, ...
             1 / ( 4 * ( n + 1 ) ) * ( ( b - a ) / n )^( n + 1 ) * df_max ];
end
err_estimated
%   err_estimated =
%     11.1033     8.7205     6.0881     3.6310     1.8689     0.8427     0.3375
plot( n_vect, err, '-ko', n_vect, err_estimated, '--ks' );
```

We verify that $e_n(f) \leq \widetilde{e}_n(f)$ for all n. Since $\lim_{n\to\infty} \widetilde{e}_n(f) = 0$, we have that $\lim_{n\to\infty} e_n(f) = 0$, i.e. the polynomial $\Pi_n f(x)$ converges to $f(x)$ as n increases, for all $x \in I$.

Solution III (Theoretical)

a) The interpolating polynomial of degree n for $f(x)$ is $\Pi_n f(x) = \sum_{k=0}^{n} f(x_k)\,\varphi_k(x)$, where $\varphi_k(x) := \prod_{i=0,\,i\neq k}^{n} \frac{x-x_i}{x_k-x_i}$ are the Lagrange characteristic functions and x_i are distinct nodes. For $n=2$, we calculate $\varphi_k(x)$ for $k=0,1,2$ as:

$$\begin{aligned}
\varphi_0(x) &= \frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2} = x^2 - \frac{5}{2}x + 1,\\
\varphi_1(x) &= \frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2} = -\frac{4}{3}x^2 + \frac{8}{3}x,\\
\varphi_2(x) &= \frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1} = \frac{1}{3}x^2 - \frac{1}{6}x.
\end{aligned}$$

By observing that $f(x_0) = -2$, $f(x_1) = -\frac{11}{8}$, and $f(x_2) = 2$, we obtain $\Pi_2 f(x) = \frac{1}{2}x^2 + x - 2$.

b) In this case, we have $\varphi_0(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$, $\varphi_1(x) = -x^2 + 2x$, and $\varphi_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x$. By observing that $f(x_0) = -2$, $f(x_1) = 0$, and $f(x_2) = 2$, we obtain $\Pi_2 f(x) = 2x - 2$ which is a polynomial of degree 1. The result is due to the fact that the data points $\{(x_i, f(x_i))\}_{i=0}^{n}$ are aligned on a straight line.

c) It is sufficient to observe that $f(x)$ is polynomial of degree 3 to conclude that $\Pi_3 f(x) \equiv f(x)$.

Solution IV (MATLAB)

a) We execute the following commands to compare the interpolating polynomials $\Pi_n f(x)$ with $f(x)$ in Figure 3:

```
f = @(x) 1 ./ ( 1 + x.^2 );    a = -5;    b = 5;
n_vect = [ 2 4 8 12 ];
x_values = linspace( a, b, 1001 );
f_values = f( x_values );
for n = n_vect
   x_nodes = linspace( a, b, n + 1 );
   y_nodes = f( x_nodes );
   P = polyfit( x_nodes, y_nodes, n );
   P_values = polyval( P, x_values );
   figure( n );
   plot( x_values, P_values, '-k', ...
            x_values, f_values, '--k', x_nodes, y_nodes, 'xk' );
   legend( '\Pi_n f(x)', 'f(x)', '(x_i,y_i)');
end
```

We observe that oscillations of the polynomials $\Pi_n f(x)$ appear at the endpoints of the interval I for "large" n, thus highlighting the so-called Runge phenomenon; the amplitude of these oscillations increases with n.

b) We plot the error $e_n(f)$ vs. n in Figure 4 with the following commands:

```
err = [ ];
for n = n_vect
   x_nodes = linspace( a, b, n + 1 );
   y_nodes = f( x_nodes );
```

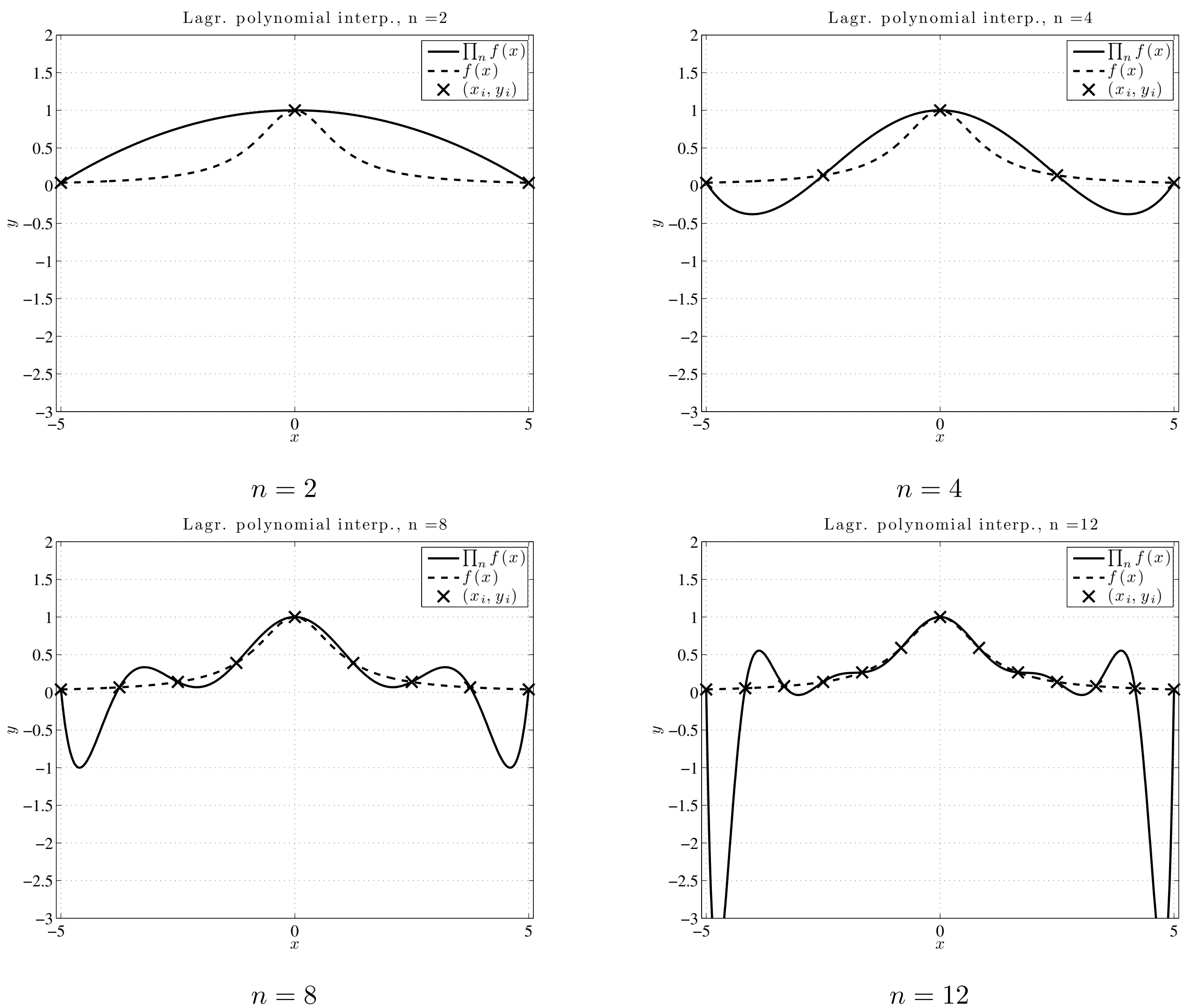

Figure 3: Interpolating polynomials $\Pi_n f(x)$ of the function $f(x) = \frac{1}{1+x^2}$ at uniformly spaced nodes in $I = [-5, 5]$ for $n = 2, 4, 8$, and 12.

```
    P = polyfit( x_nodes, y_nodes, n );
    P_values = polyval( P, x_values );
    err = [ err, max( abs( P_values - f_values ) ) ];
end
err
%  err =
%      0.6462    0.4384    1.0452    3.6630
figure; plot( n_vect, err, '-ko' );
```

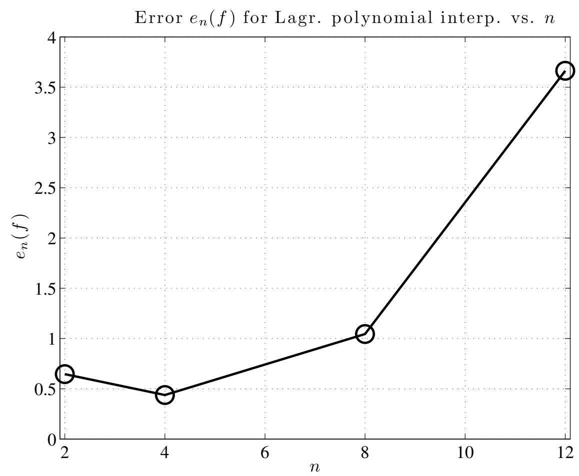

Figure 4: Errors $e_n(f)$ vs. n for interpolating polynomials $\Pi_n f(x)$ of the function $f(x) = \frac{1}{1+x^2}$ at uniformly spaced nodes in $I = [-5, 5]$.

As already observed in point a), we note that the error $e_n(f)$ increases for increasing n.

c) We repeat point a) by using the Chebyshev-Gauss-Lobatto nodes, and we denote the corresponding interpolating polynomials by $\Pi_n^c f(x)$. In MATLAB, we use the following commands to obtain the results in Figure 5:

```
for n = n_vect
    x_nodes_c = (a+b)/2 + (b-a)/2 * ( - cos( pi * [ 0 : n ] / n ) );
    y_nodes_c = f( x_nodes_c );
    P_c = polyfit( x_nodes_c, y_nodes_c, n );
    P_c_values = polyval( P_c, x_values );
    figure( n + 100);
    plot( x_values, P_c_values, '-k', ...
          x_values, f_values, '--k', x_nodes_c, y_nodes_c, 'xk' );
    legend( '$\prod_n f(x)$', '$f(x)$', '$(x_i,y_i)$' );
end
```

We observe that the interpolating polynomials $\Pi_n^c f(x)$ converge to $f(x)$ for increasing values of n. In Figure 6 we compare the interpolating polynomials $\Pi_8^c f(x)$ and $\Pi_8 f(x)$ with $f(x)$.

d) By repeating point b) for the Chebyshev-Gauss-Lobatto nodes, we obtain that the error $e_n^c(f)$ associated to $\Pi_n^c f(x)$ decreases for increasing values of n (see Figure 7). We use the following MATLAB commands:

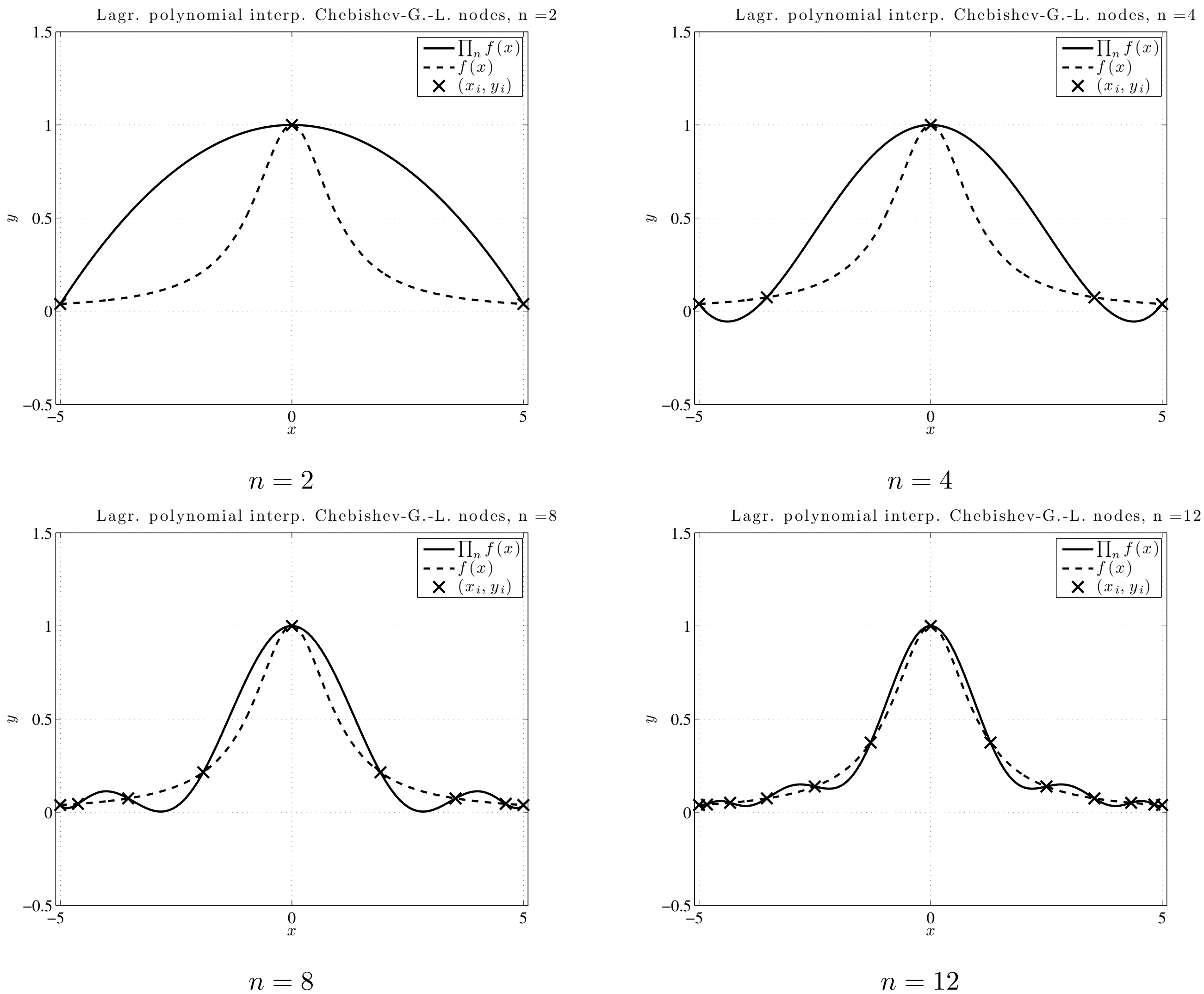

Figure 5: Interpolating polynomials $\Pi_n f(x)$ of the function $f(x) = \frac{1}{1+x^2}$ at the Chebyshev-Gauss-Lobatto nodes in $I = [-5, 5]$ for $n = 2, 4, 8$, and 12.

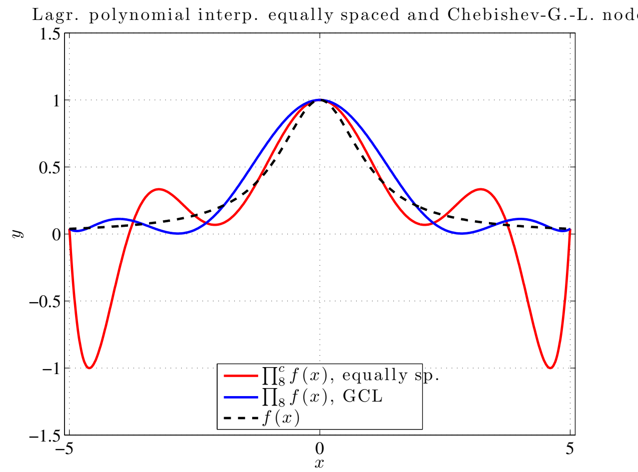

Figure 6: Interpolating polynomials $\Pi_8^c f(x)$ and $\Pi_8 f(x)$ of the function $f(x) = \frac{1}{1+x^2}$ at the Chebyshev-Gauss-Lobatto and uniformly spaced nodes in $I = [-5, 5]$, respectively.

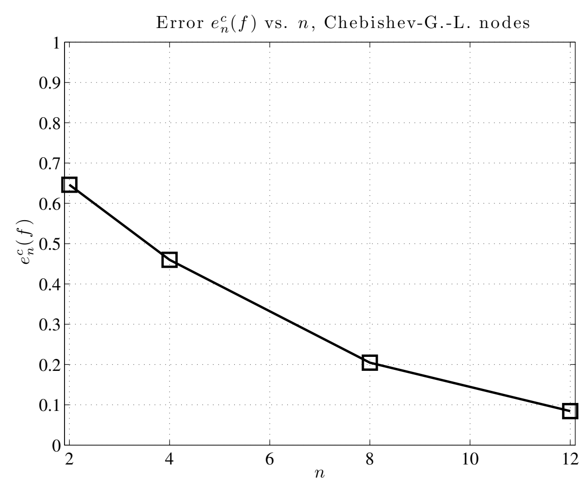

Figure 7: Errors $e_n^c(f)$ vs. n for interpolating polynomials $\Pi_n^c f(x)$ of the function $f(x) = \frac{1}{1+x^2}$ at the Chebyshev-Gauss-Lobatto nodes in $I = [-5, 5]$; $n = 2, 4, 8$, and 12.

```
err_c = [ ];
for n = n_vect
   x_nodes_c = (a+b)/2 + (b-a)/2 * ( - cos( pi * [ 0 : n ] / n ) );
   y_nodes_c = f( x_nodes_c );
   P_c = polyfit( x_nodes_c, y_nodes_c, n );
   P_values_c = polyval( P_c, x_values );
   err_c = [ err_c, max( abs( P_values_c - f_values ) ) ];
end
err_c
%  err_c =
%     6.4623e-01   4.5998e-01   2.0468e-01   8.4396e-02
plot( n_vect, err_c, '-ks'  );
```

The result is justified by the fact that the use of the Chebyshev-Gauss-Lobatto nodes ensures that $\lim_{n\to\infty} e_n^c(f) = 0$ for $f(x) \in C^\infty(I)$.

Solution V (Theoretical)

a) In general, given a function $f(x) \in C^{n+1}(I)$ with $I = [a, b]$ and the corresponding interpolating polynomial $\Pi_n f(x)$ at uniformly spaced nodes $\{x_i\}_{i=0}^n$, we have the following estimate for the error $e_n(f) := \max_{x\in I} |f(x) - \Pi_n f(x)|$:

$$e_n(f) \leq \widetilde{e}_n(f) = \frac{1}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1} \max_{x\in I} \left|f^{(n+1)}(x)\right|.$$

Specifically, for $f(x) = \sin\left(\frac{x}{3}\right)$, we obtain that $f^{(1)}(x) = \frac{1}{3}\cos\left(\frac{x}{3}\right)$, $f^{(2)}(x) = -\frac{1}{9}\sin\left(\frac{x}{3}\right)$, $f^{(3)}(x) = -\frac{1}{27}\cos\left(\frac{x}{3}\right)$, ...; as consequence, since $I = [a, b] = [0, 1]$, we deduce that $\max_{x\in I} \left|f^{(n+1)}(x)\right| \leq \frac{1}{3^{n+1}}$. By the previous result, we obtain that:

$$e_n(f) \leq \widetilde{e}_n(f) = \frac{1}{4(n+1)(3n)^{n+1}}.$$

Since $\lim_{n\to\infty} \widetilde{e}_n(f) = 0$, we conclude that the error $e_n(f)$ tends to zero as n increases.

b) We proceed by trial-and-error, evaluating $\widetilde{e}_n(f)$ for $n = 1, 2, 3, \ldots$. We obtain $\widetilde{e}_1(f) = 1.3889 \cdot 10^{-2}$, $\widetilde{e}_2(f) = 3.8580 \cdot 10^{-4}$, and, finally, $\widetilde{e}_3(f) = 9.5260 \cdot 10^{-6}$. As a consequence, the minimum number of equally spaced nodes in I necessary to ensure that $e_n(f) < 10^{-4}$ is $n + 1 = 4$.

c) The Chebyshev-Gauss-Lobatto nodes in $I = [a, b]$ are determined by the formula

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\,\widehat{x}_i, \qquad \text{where } \widehat{x}_i := -\cos\left(\frac{\pi}{n} i\right), \qquad \text{for } i = 0, \ldots, n.$$

For $n = 3$, we have $\widehat{x}_0 = -1$, $\widehat{x}_1 = -\frac{1}{2}$, $\widehat{x}_2 = \frac{1}{2}$, $\widehat{x}_2 = 1$. Since $a = 0$ and $b = 1$, we obtain $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $x_2 = 1$.

d) Since the Chebyshev-Gauss-Lobatto nodes are not uniformly spaced in I, we consider the following error estimate for the interpolating polynomials $\Pi_n f(x)$:

$$e_n(f) \leq \widetilde{e}_n(f) = \frac{1}{(n+1)!} \max_{x\in I} \left|f^{(n+1)}(x)\right| \max_{x\in I} |\omega_n(x)|.$$

We observe that $\max_{x\in I} |\omega_3(x)| < 0.016$ (from Figure 1 of the exercise sheet) and $\max_{x\in I} \left|f^{(4)}(x)\right| \leq \frac{1}{3^4}$ from point a). We obtain that $e_3(f) \leq \widetilde{e}_3(f) = 8.2305 \cdot 10^{-6}$.