Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Solutions – Nonlinear equations: fixed point iterations

Solution I (MATLAB)

a) We consider the following implementation of the function `fixed_point_iterations.m`:

```
function [xvect, nit] = fixed_point_iterations( phi, x0, tol, nmax )
% FIXED_POINT_ITERATIONS Finds a fixed point of a scalar function.
%   [XVECT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) finds a fixed point of
%   the iteration function PHI using the fixed point iterations method and
%   returns a vector XVECT containing the successive approximations of the
%   fixed point (iterates).
%   PHI accepts a real scalar input x and returns a real scalar value;
%   PHI can also be an inline object. X0 is the initial guess.
%   TOL is the tolerance on error allowed and NMAX the maximum number of
%   iterations.
%   The stopping criterion based on the difference of successive iterates is used.
%   If the search fails a warning message is displayed.
%
%   [XVECT,NIT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) also returns the
%   number of iterations  NIT.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

nit = 0;
xvect(nit+1) = x0;
err_estim = tol + 1;
while ( err_estim ≥ tol && nit < nmax )
    xvect(nit+2) = phi( xvect(nit+1) );
    err_estim = abs( xvect(nit+2) - xvect(nit+1) ); % diff. successive iterates
    nit = nit + 1;
end

if err_estim ≥ tol
    warning(['Fixed point iter. stopped without converging to the desired '...
             'tolerance, the maximum number of iterations was reached.']);
end
```

```
return
```

We consider two choices of the initial guess $x^{(0)} = -\pi/4$ and $x^{(0)} = \pi/5$. For $x^{(0)} = -\pi/4$ we obtain $x^{(k_c)} = 0.7390854\ldots$. For $x^{(0)} = \pi/5$ we get $x^{(k_c)} = 0.7390847\ldots$.

b) We obtain that $k_c = 30$ and $e^{(k_c)} = 3.3407 \cdot 10^{-7}$ for the case $x^{(0)} = -\pi/4$. Instead, for $x^{(0)} = \pi/5$, we obtain $k_c = 32$ and $e^{(k_c)} = 3.4167 \cdot 10^{-7}$. We use the following MATLAB commands:

```
phi = @(x) cos(x);
tol = 1e-6;
kmax = 1500;
alpha = 0.739085133215161;
x0 = -pi/4;
[xvect, kc] = fixed_point_iterations( phi, x0, tol, kmax );
errvect = abs( xvect - alpha );
kc, err = errvect( end )
% kc =
%
%      30
%
% err =
%
%    3.3407e-07
```

c) We plot the error in semi-logarithmic scale by means of the following MATLAB commands:

```
kvect = 0 : kc;
figure( 1 ); semilogy( kvect, errvect, '-ok' ); grid on; legend('x^{(0)}=-\pi/4');
```

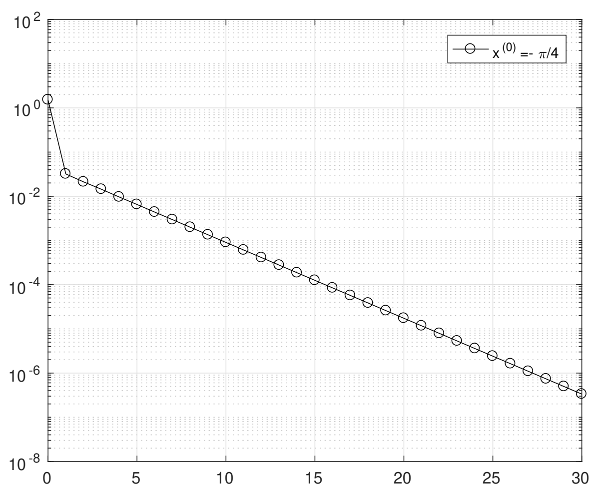

Figure 1: $e^{(k)}$ vs k with $x^{(0)} = -\pi/4$.

The results for the case $x^{(0)} = -\pi/4$ are depicted in Figure 1. We can observe graphically the linear convergence of the fixed point iterations algorithm to α starting from the inital guess

$x^{(0)} = -\pi/4$. Similar results can be obtained for the case $x^{(0)} = \pi/5$.

Solution II (Theoretical and MATLAB)

a) We recall that α is a fixed point of $\phi(x)$ iff $\phi(\alpha) = \alpha$. From this, we can verify that α_2 is indeed a fixed point of $\phi(x)$. Moreover, we verify that the iteration function $\phi(x)$ is continuously differentiable on I_2 (in fact, $\phi \in C^\infty(\mathbb{R})$). Then, we recall the following results for the global and local convergence of the fixed point iterations algorithm valid for the case of continuous differentiable iteration functions.

Proposition 1 (global (Proposition 2.8 in the lecture notes)) *If the iteration function $\phi(x)$ is such that $\phi \in C^1([a,b])$, $\phi(x) \in [a,b]$ for all $x \in [a,b]$, and $|\phi'(x)| < 1$ for all $x \in [a,b]$, then there exists a unique fixed point α of $\phi(x)$ in the interval $[a,b]$ and the fixed point iterations algorithm is convergent to α ($x^{(k)} \to \alpha$ for $k \to \infty$) for all the initial values $x^{(0)} \in [a,b]$; moreover, the algorithm is at least linearly convergent to α (order 1), i.e.:*

$$\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha).$$

Proposition 2 (local, Ostrowski's theorem (Proposition 2.10 in the lecture notes)) *Let α be a fixed point of the iteration function $\phi(x)$. If $\phi \in C^1(I_\alpha)$, with I_α a neighborhood of α, and $|\phi'(\alpha)| < 1$, then, for $x^{(0)}$ sufficiently close to α, the fixed point iterations algorithm is convergent to α ($x^{(k)} \to \alpha$ for $k \to \infty$); moreover, the algorithm is at least linearly convergent to α (order 1), i.e.:*

$$\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha),$$

with the asymptotic convergence factor $\mu = \phi'(\alpha)$.

We plot $\phi(x)$ and $\phi'(x)$ for $x \in I_2$ in Figure 2. We verify the existence and uniqueness of the fixed point $\alpha_2 \in I_2 = [a_2, b_2] = [\pi/2, \pi]$ ($\phi(\alpha_2) = \alpha_2$).

```
phi = @(x) x/2 + sin(x) - pi/6 + sqrt(3)/2;
dphi = @(x) 1/2 + cos(x);
a2 = pi/2; b2 = pi;
xv = linspace( a2, b2, 1001 );
figure( 1 ); plot( xv, phi( xv ), '-k', xv, xv, '--k' ); grid on
axis equal; axis([ a2 b2 a2 b2 ]);
figure( 2 ); plot( xv, dphi( xv ), '-k', xv, -ones( 1, length( xv ) ), ...
     '--k', xv, ones( 1, length( xv ) ), '-.k' ); axis([ a2 b2 -1.1 1.1 ]);
alpha2 = 2.246005589297;
phi( alpha2 ) - alpha2
dphi( alpha2 )
% ans =
%
%    1.0960e-12
%
% ans =
%
%    -0.1251
```

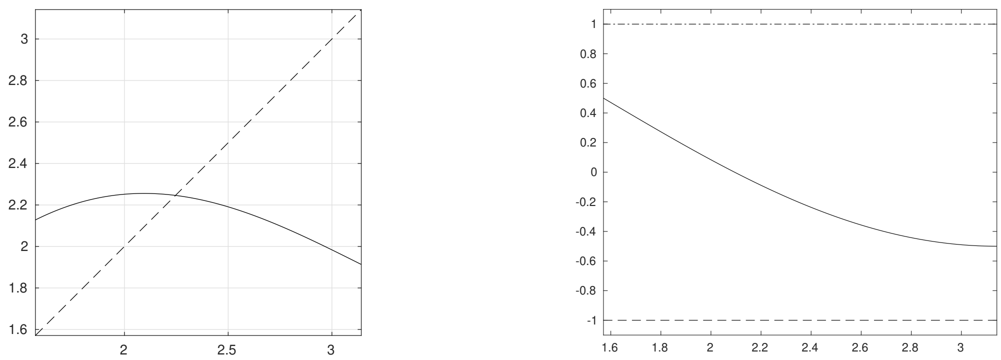

Figure 2: $\phi(x)$ (left) and $\phi'(x)$ (right) for $x \in I_2$.

We verify that $\phi(x) \in I_2$ (i.e. $a_2 \leq \phi(x) \leq b_2$) for all $x \in I_2$ and that $|\phi'(x)| < 1$ for all $x \in I_2$. By Proposition 1, the fixed point iterations algorithm is globally convergent to α_2 over I_2. The fixed point iterations are also locally convergent to α in the sense of Proposition 2, with convergence order equal to one (linear convergence) and the asymptotic convergence factor $\mu = \phi'(\alpha_2) = -0.1251$.

b) We consider two choices of the initial guess $x^{(0)} = \pi/2$ and $x^{(0)} = \pi$. For $x^{(0)} = \pi/2$:

```
phi = @(x) x/2 + sin(x) - pi/6 + sqrt(3)/2;
tol = 1e-6;
kmax = 1500;
alpha2 = 2.246005589297;
x0 = pi/2;
[x2vect, kc2] = fixed_point_iterations( phi, x0, tol, kmax );
err2vect = abs( x2vect - alpha2 );
kc2, err2 = err2vect( end )
% kc2 =
%
%      8
%
% err2 =
%
%    3.5940e-08
```

We obtain that $k_c = 8$ and $e^{(k_c)} = 3.5940 \cdot 10^{-8}$. By repeating the calculation for $x^{(0)} = \pi$, we obtain $k_c = 8$ and $e^{(k_c)} = 1.8954 \cdot 10^{-8}$. We plot the errors $e^{(k)}$ vs. k and the ratios $a^{(k)}$ vs k, for both $x^{(0)} = \pi/2$ and $x^{(0)} = \pi$. For instance, for $x^{(0)} = \pi/2$:

```
k2vect = 0 : kc2;
figure( 1 ); semilogy( k2vect, err2vect, '-ok' );
ak2vect = ( x2vect( 2 : end ) - alpha2 ) ./ ( x2vect( 1 : end - 1 ) - alpha2 );
figure( 2 ); plot( k2vect( 1 : end - 1 ), ak2vect, '-ok' );
```

We obtain the results reported in Figure 3. We deduce the linear convergence of the fixed point iterations algorithm to α_2 with the asymptotic convergence factor $\mu = \phi'(\alpha_2) = -0.1251$. The numerical results confirm the what was discussed in point a).

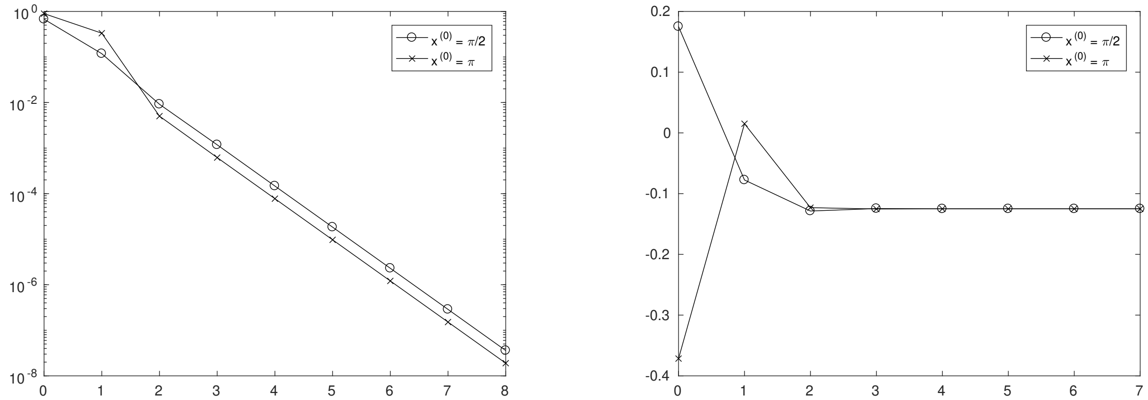

Figure 3: $e^{(k)}$ (left) and $a^{(k)}$ (right) vs k for α_2, with $x^{(0)} = \pi/2$ and $x^{(0)} = \pi$.

c) We repeat point a) for $\alpha_1 \in I_1$ by plotting $\phi(x)$ and $\phi'(x)$ for $x \in I_1 = [a_1, b_1] = -[\pi/2, 0]$ in Figure 4. We verify the existence and uniqueness of the fixed point $\alpha_1 \in I_1$ ($\phi(\alpha_1) = \alpha_1$). In the plot, we see that two of the hypotheses of Proposition 1 are violated, since there exists $x \in [a_1, b_1]$ such that $\phi(x) > b_1$ and $\phi'(x) \geq 1$. For these reasons, the global convergence to α_1 of the fixed point iterations algorithm is not guaranteed for all the choices of the initial value $x^{(0)} \in I_1$ and we need to study the local convergence properties of the method by means of Proposition 2. However, we observe that the hypothesis on the derivative of $\phi(x)$ at the fixed point α_1 is not satisfied, since $\phi'(\alpha_1) = 1$. For this reason, Proposition 2 cannot be used and, in general, we cannot guarantee the convergence of the fixed point iterations to α_1 even for an initial value $x^{(0)}$ sufficiently close to α_1.

Nevertheless, since the special case $\phi'(\alpha_1) = 1$ occurs, the convergence of the fixed point iterations to α_1 depends on the properties of the iteration function $\phi(x)$ in a neighborhood of α_1 and on the choice of $x^{(0)}$. For example, we can deduce from Figure 4 (left) that the algorithm converges to α_1 for $x^{(0)} \in [a_1, \alpha_1]$, but diverges if $x^{(0)} \in (\alpha_1, b_1]$.

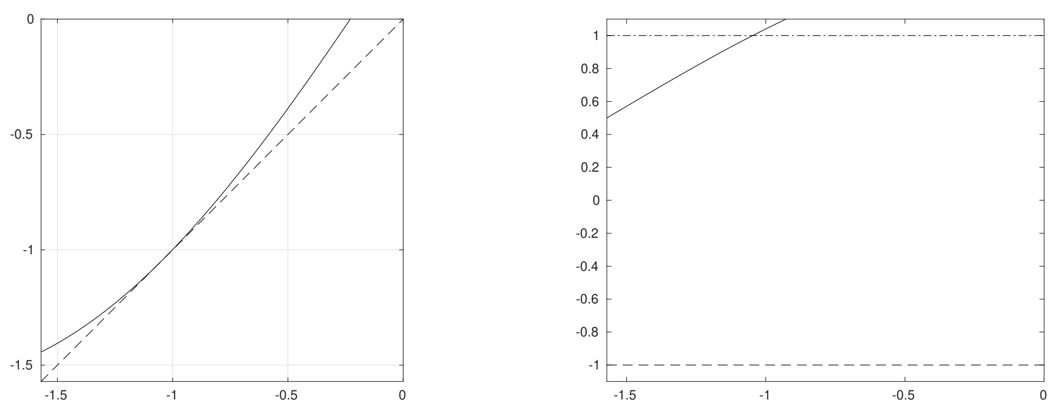

Figure 4: $\phi(x)$ (left) and $\phi'(x)$ (right) for $x \in I_1$.

d) By repeating point b) for the fixed point α_1, we verify that, as expected, the fixed point algorithm for $x^{(0)} = -0.9$ does not converge to α_1 (incidentally, for this choice of $x^{(0)}$, the algorithm converges to α_2). For $x^{(0)} = -1.1$ the algorithm converges to α_1 in $k_c = 1473$ iterations with the error $e^{(k_c)} = 1.5174 \cdot 10^{-3}$. We report in Figure 5 the errors $e^{(k)}$ and the ratios $a^{(k)}$ vs k for the latter case. We notice that the convergence order to α_1 is less than linear and the ratio $a^{(k)}$ tends to 1 for large k. This justifies the slow convergence of the algorithm.

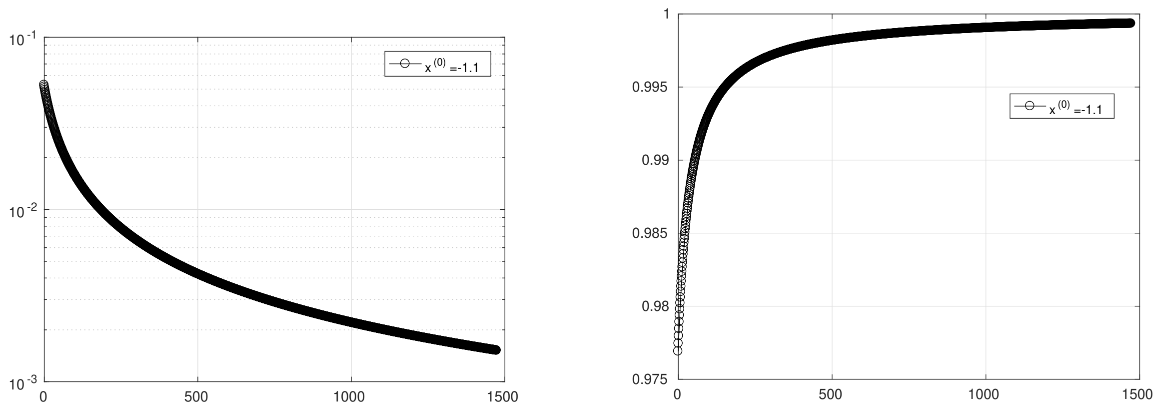

Figure 5: $e^{(k)}$ (left) and $a^{(k)}$ (right) vs k for α_2, with $x^{(0)} = -1.1$.

e) Since $\phi \in C^1(I_2)$, by using the Lagrange theorem (mean value theorem), there exists $\xi^{(k)}$ between $x^{(k)}$ and α_2 such that $x^{(k+1)} - \alpha_2 = \phi(x^{(k)}) - \phi(\alpha_2) = \phi'(\xi^{(k)})(x^{(k)} - \alpha_2)$ for all $k \geq 0$. We deduce that:

$$|x^{(k+1)} - \alpha_2| \leq \max_{x \in I_2} |\phi'(x)||x^{(k)} - \alpha_2|, \quad \text{for all } k \geq 0,$$

so that $C = \max_{x \in I_2} |\phi'(x)|$. From Figure 2 (right) or by using the MATLAB commands below, we can deduce that $C = 1/2$ (we can also use the expression of $\phi'(x) = 1/2 + \cos(x)$, where $x \in I_2 = [\pi/2, \pi]$, which in turn implies $|\phi'(x)| \leq 1/2$).

```
xv = linspace( a2, b2, 1001 );
dphi_max_I2 = max( abs( dphi( xv ) ) )
% dphi_max_I2 =
%
%     0.5000
```

f) From point e), we obtain by recursion that:

$$|x^{(k)} - \alpha_2| \leq C|x^{(k-1)} - \alpha_2| \leq \ldots \leq C^k|x^{(0)} - \alpha_2|, \quad \text{for all } k \geq 0.$$

We seek the minimum number of iterations k_{min} for which we can guarantee an error $|x^{(k_{min})} - \alpha_2|$ smaller than the tolerance 2^{-20} for all $x^{(0)} \in I_2$. To this aim, we observe that $|x^{(0)} - \alpha_2| \leq |I_2|$, for all $\alpha_2 \in I_2$, and we solve the following inequality:

$$C^{k_{min}}|x^{(0)} - \alpha_2| \leq C^{k_{min}}|I_2| < tol.$$

Since $C = 1/2$, we need $k_{min} > \log(tol/|I_2|)/\log C = 20.6515$, which means $k_{min} = 21$. Note that, had α_2 been known, we could have used the value $\max_{x^{(0)} \in I_2} |x^{(0)} - \alpha_2| = \pi - \alpha_2$ instead of $|I_2|$ in the previous formula. This yields $k_{min} = 20$.

g) Since $\phi \in C^1(\mathbb{R})$, by using the Lagrange theorem as in point e), there exists $\xi^{(k)}$ between $x^{(k)}$ and α ($\alpha = \alpha_1$ or α_2) such that $x^{(k+1)} - \alpha = \phi'(\xi^{(k)})(x^{(k)} - \alpha)$ for all $k \geq 0$. By adding and subtracting $x^{(k)}$ on the left hand side of the previous relation, we obtain:

$$\alpha - x^{(k)} = \frac{1}{1 - \phi'(\xi^{(k)})}(x^{(k+1)} - x^{(k)}), \quad \text{for all } k \geq 0.$$

For $\alpha = \alpha_2$, we have that $\phi'(\xi^{(k)}) \simeq \phi'(\alpha_2) = -0.1251$, so that $|x^{(k)} - \alpha_2| \simeq 0.8888|x^{(k+1)} - x^{(k)}|$ for $k \to \infty$. Therefore, the stopping criterion based on the increment of successive approximate zeros is satisfactory for the fixed point α_2 (the error is only slightly overestimated).

For $\alpha = \alpha_1$, we have that $\phi'(\xi^{(k)}) \simeq \phi'(\alpha_1) = 1$, so that $|x^{(k)} - \alpha_1| \simeq M|x^{(k+1)} - x^{(k)}|$ with M large ($M \to \infty$) for $k \to \infty$. Provided that the fixed point iterations algorithm converges to the fixed point α_1, the stopping criterion based on the increment of successive approximate zeros is not satisfactory for α_1 and the error is largely underestimated.

We verify these results numerically with MATLAB. Starting from $x^{(0)} = \pi/2$ for the fixed point α_2, we verify that the stopping criterion based on the difference of successive approximate zeros is satisfactory and slightly overestimating the error. Indeed, as predicted, we obtain $e^{(k_c-1)}/|x^{(k_c)} - x^{(k_c-1)}| = M_2 = 0.8888$ with the following command:

```
M2 = err2vect( end - 1 ) / abs( x2vect( end ) - x2vect( end - 1 ) )
% M2 =
%
%      0.8888
```

By repeating for $x^{(0)} = -1.1$ for the fixed point α_1, we obtain $e^{(k_c-1)}/|x^{(k_c)} - x^{(k_c-1)}| = M_1 = 1520.4$. We conclude that the stopping criterion is not satisfactory and the error is largely underestimated by the difference of successive approximate zeros.

Solution III (Theoretical)

a) We observe that $f(x) \in C^0(I)$, with $I = [0.02, 0.2]$. We can verify, also by using MATLAB, that $f(0.02) = -0.555\ldots$ and $f(0.2) = 0.563\ldots$, so that there is a change of sign in the interval. We deduce that there exists at least one zero $\alpha \in I$. Moreover, the zero α is unique, since the function $f(x)$ is monotonically increasing in the interval ($f'(x) > 0$ over the interval), as we can verify by plotting the function $f(x)$ in MATLAB (Figure 6).

b) First, we need to verify that the zero α of $f(x)$ is a fixed point of $\phi_1(x)$ and $\phi_2(x)$. We observe that $\phi_1(\alpha) = \log(2 - 3\sqrt{\alpha}) = \log(e^\alpha) = \alpha$ (we deduce that $2 - 3\sqrt{\alpha} = e^\alpha$ from $f(\alpha) = 0$); similarly, $\phi_2(\alpha) = (2 - e^\alpha)^2/9 = (2 - (2 - 3\sqrt{\alpha}))^2/9 = \alpha$. As a consequence, both iteration functions admit a fixed point α corresponding to the zero of the function $f(x)$. Both iteration functions ϕ_1, ϕ_2 are continuously differentiable over I. We plot them and their derivatives $\phi_1'(x) = 3/(2\sqrt{x}\,(3\sqrt{x} - 2))$ and $\phi_2'(x) = (2e^x\,(e^x - 2))/9$ in Figures 7 and 8, respectively.

We observe that iteration function $\phi_1(x)$ violates the hypotheses of Proposition 1 (there exist values $x \in [0.02, 0.2]$ such that $\phi_1(x) < 0.02$ or $\phi_1(x) > 0.2$, and $|\phi_1'(x)| > 1$). Moreover, Proposition 2 cannot be used since $|\phi_1'(\alpha)| > 1$. Specifically, the fixed point iterations algorithm cannot converge to α for any $x^{(0)} \neq \alpha$, since $|\phi_1'(\alpha)| > 1$.

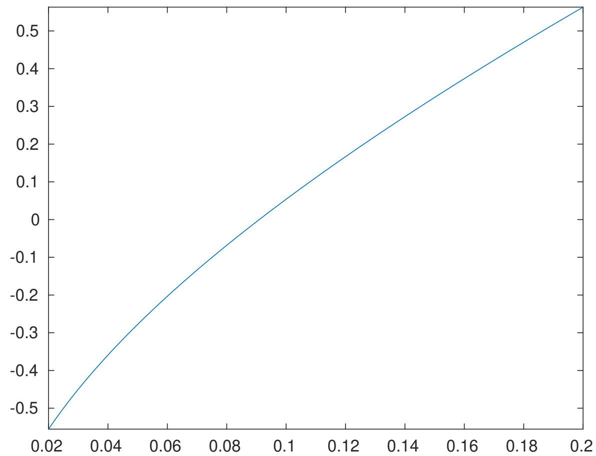

Figure 6: Function $f(x)$ for $x \in I$.

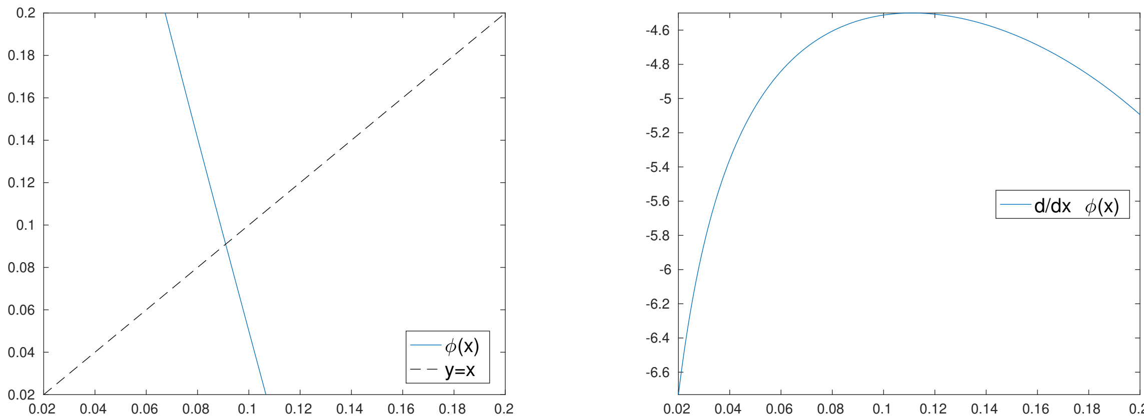

Figure 7: Iteration function $\phi_1(x)$ (left) and its derivative $\phi_1'(x)$ (right) for $x \in I$.

When considering $\phi_2(x)$, we observe that all the hypotheses of Proposition 1 are satisfied, so that the fixed point iterations algorithm is globally convergent to α for all $x^{(0)} \in I = [0.02, 0.2]$. The algorithm is also locally and linearly convergent to α with the asymptotic convergence factor $\phi_2'(\alpha) \neq 0$.

For these reasons, we would select $\phi_2(x)$ as iteration function to find the zero $\alpha \in I = [0.02, 0.2]$ of $f(x)$.

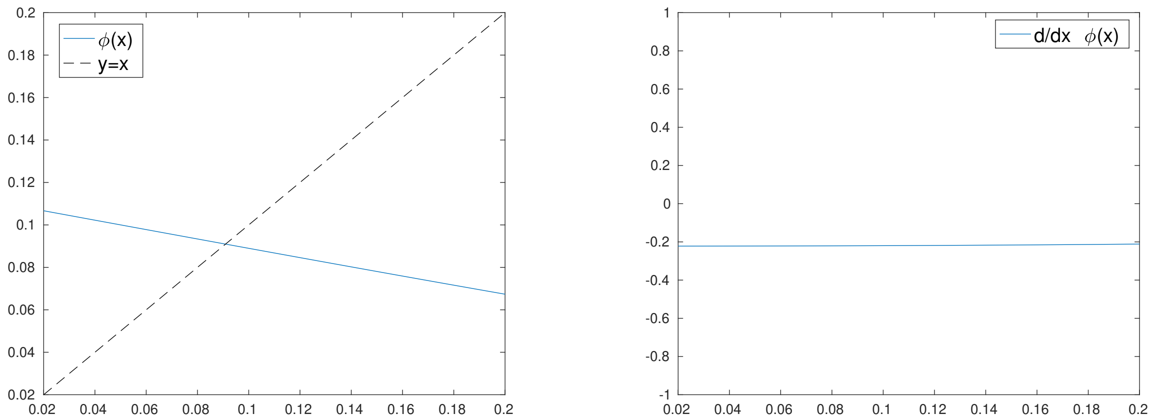

Figure 8: Iteration function $\phi_2(x)$ (left) and its derivative $\phi_2'(x)$ (right) for $x \in I$.

c) The first approximate zeros for $\phi_1(x)$ are $x^{(1)} = 0.2846$ and $x^{(2)} = -0.9171$. This clearly highlights the divergence of the algorithm. For $\phi_2(x)$ we have $x^{(1)} = 0.1000$ and $x^{(2)} = 0.0890$, converging towards $\alpha \simeq 0.0910$. In MATLAB, e.g. for $\phi_1(x)$:

```
phi1 = @(x) log( 2 - 3 * sqrt(x) );
x = 0.05;
for i = 1 : 2
    x = phi1( x )
end
```

In Figures 9 and 10 we graphically highlight the first fixed points iterations for $\phi_1(x)$ and $\phi_2(x)$.

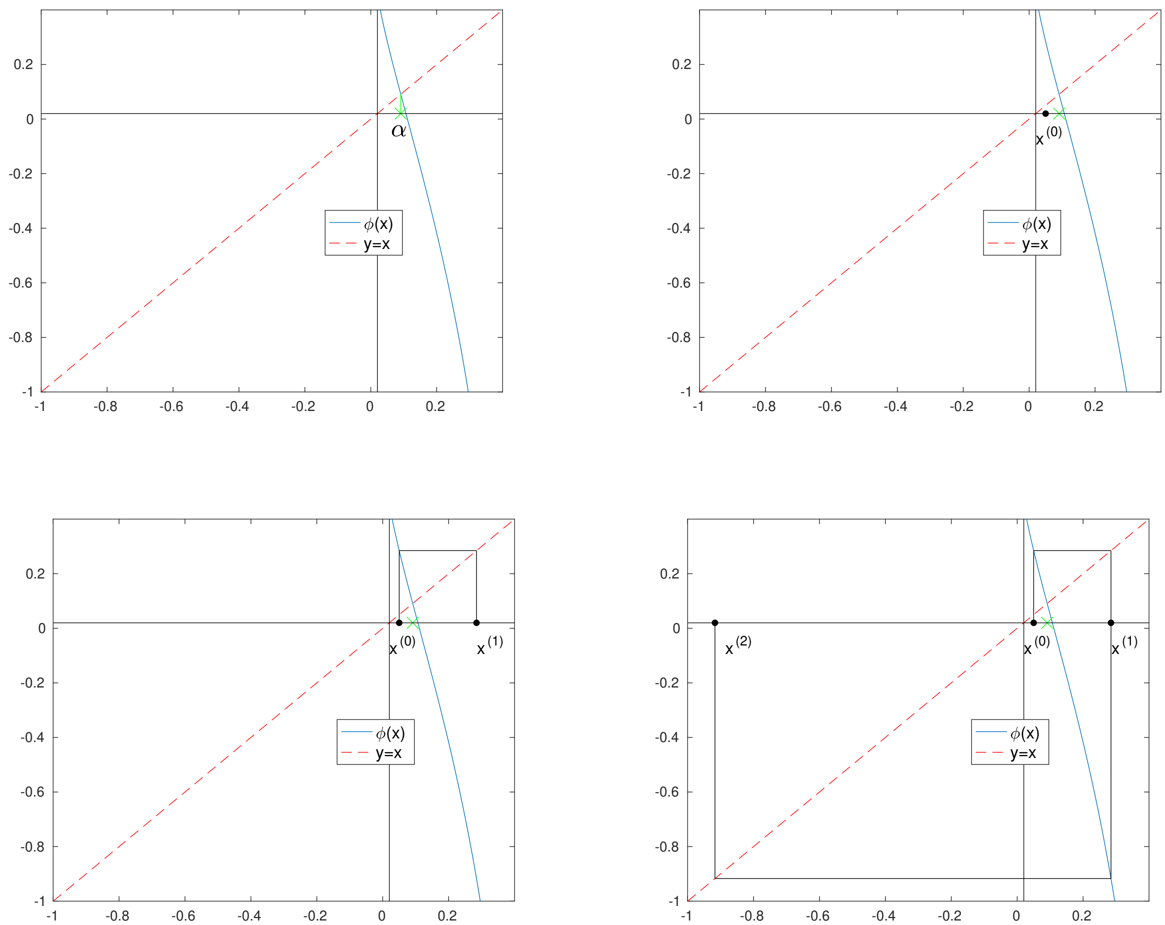

Figure 9: Fixed point iterations for $\phi_1(x)$ with $x^{(0)} = 0.05$.

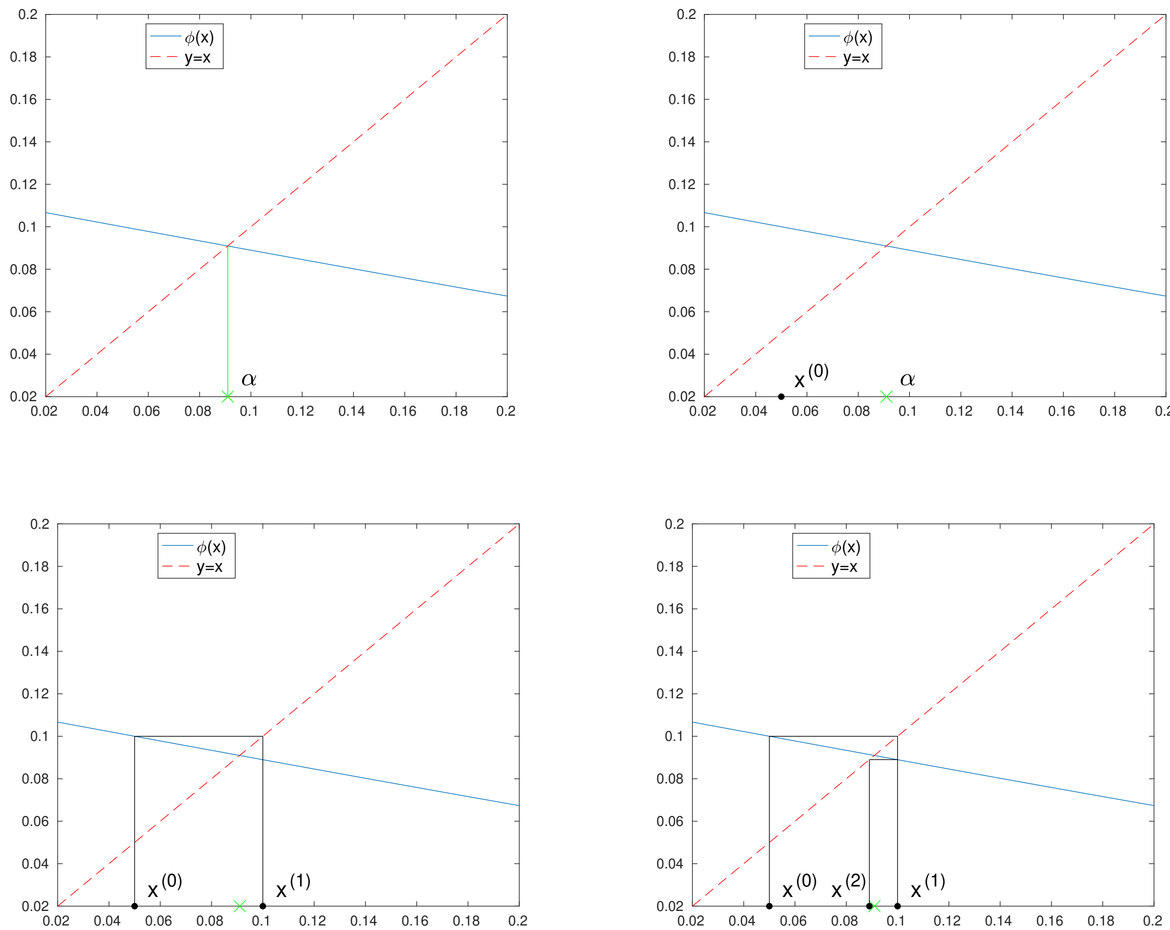

Figure 10: Fixed point iterations for $\phi_2(x)$ with $x^{(0)} = 0.05$.

Solution IV (OPTIONAL, Theoretical)

a) The k-th iteration of the Newton method corresponds to $x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(x^{(k)})$, for all $k \geq 0$, if $f'(x^{(k)}) \neq 0$, for some initial guess $x^{(0)}$. If we introduce the iteration function $\phi_N(x) = x - f(x)/f'(x)$, then the Newton method can be recast as a fixed point iterations algorithm whose k-th iteration reads $x^{(k+1)} = \phi_N(x^{(k)})$.

b) By setting $f(x) = (x-\alpha)^m g(x)$ in a neighborhood of I_α of α, with $g(\alpha) \neq 0$, we obtain: $f'(x) = (x-\alpha)^{m-1}[m\, g(x) + (x-\alpha)\, g'(x)]$ and $f''(x) = (x-\alpha)^{m-2}[m\,(m-1)\, g(x) + 2m(x-\alpha)\, g'(x) + (x-\alpha)^2\, g''(x)]$. Then, we observe that:

$$\phi_N'(x) = 1 - \frac{f'(x)^2 - f(x)\, f''(x)}{f'(x)^2} = 1 - \frac{m\, g^2(x) + (x-\alpha)^2[(g'(x))^2 - g(x)g''(x)]}{m^2\, g^2(x) + 2m(x-\alpha)g(x)g'(x) + (x-\alpha)^2(g'(x))^2}.$$

For $x = \alpha$ we obtain:

$$\phi_N'(\alpha) = 1 - \frac{1}{m},$$

since $g(\alpha) \neq 0$.

c) We recall Proposition 2 and the following result.

Proposition 3 (local) *By assuming that the hypotheses of Proposition 2 are satisfied and that, in addition, $\phi \in C^2(I_\alpha)$ with $\phi'(\alpha) = 0$ and $\phi''(\alpha) \neq 0$, then, for $x^{(0)}$ sufficiently close to α, the fixed point iterations algorithm converges quadratically to α (order 2) and:*

$$\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{(x^{(k)} - \alpha)^2} = \frac{1}{2}\phi''(\alpha).$$

When considering the Newton method with $\phi_N'(\alpha) = 1 - 1/m$, we obtain that the method is at least linearly convergent to α in virtue of Proposition 2.

If the zero α is single ($m = 1$), then the hypotheses of Proposition 3 are satisfied and the Newton method is quadratically convergent to α with the asymptotic convergence factor $\mu = \frac{1}{2}\phi_N''(\alpha) = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$; the result is obtained by observing that $\phi_N''(x) = [(f'(x))^2 f''(x) + f(x)f'(x)f'''(x) - 2f(x)(f''(x))^2]/(f'(x))^3$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ for the zero α of multiplicity $m = 1$. If the zero α is multiple ($m > 1$), then $\phi_N'(\alpha) \neq 0$ and Proposition 3 cannot be used; therefore, we only expect linear convergence of the Newton method for the zero α of multiplicity $m > 1$ according to what stated in Proposition 2.

d) The k-th iteration of the modified Newton method corresponds to $x^{(k+1)} = x^{(k)} - mf(x^{(k)})/f'(x^{(k)})$, for all $k \geq 0$, such that $f'(x^{(k)}) \neq 0$, with $m \geq 1$. The iteration function corresponding to the modified Newton method is $\phi_{N_m}'(x) = x - mf(x)/f'(x)$. By proceeding similarly to point b), we deduce that:

$$\phi_{N_m}'(\alpha) = 1 - m\frac{1}{m} = 0, \quad \text{for all } m \geq 1.$$

As consequence, following point c) and Proposition 3, the modified Newton method converges quadratically to the zero α of multiplicity $m \geq 1$, since, in general, $\phi_{N_m}''(\alpha) \neq 0$.

e) Following Exercise 2, point g), we deduce that the stopping criterion on the increment of successive approximate zeros is satisfactory for the Newton method only for the zeros α of multiplicity $m = 1$. Indeed, for $k \to \infty$:

$$|x^{(k)} - \alpha| \simeq m|x^{(k+1)} - x^{(k)}|,$$

since $\phi'_N(\xi^{(k)}) \simeq \phi'_N(\alpha) = 1 - 1/m$ in a neighborhood of α. For $m > 1$ the error is underestimated by the difference of the successive approximate zeros. for $m = 100$ the error is underestimated by two orders of magnitude.

On the contrary, when using the modified Newton method, the criterion is always satisfactory, since $\phi'_{N_m}(\alpha) = 0$ and $|x^{(k)} - \alpha| \simeq |x^{(k+1)} - x^{(k)}|$ for $k \to \infty$, independently of m.