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## Numerical Analysis and Computational Mathematics

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## Nonlinear equations: fixed point iterations

### Exercise I (MATLAB)

We aim at solving the equation  $f(x) = 0$  in  $[a, b]$ , with  $f(x) = x - \cos(x)$ ,  $a = -\pi/3$  and  $b = \pi/3$ . The function  $f(x)$  has one zero, which we denote by  $\alpha \in [a, b]$ , where  $\alpha \simeq 0.739085133215161\dots$ . We consider the fixed point iterations algorithm, with iteration function  $\phi(x) = \cos(x)$ . The generic iteration of the algorithm is:

$$x^{(k+1)} = \phi(x^{(k)}), \quad \text{for all } k \geq 0, \quad (1)$$

for a given initial value  $x^{(0)}$ .

a) With the help of the file `fixed_point_iterations_template.m`, implement the fixed point iterations method in a MATLAB function, to be saved in file `fixed_point_iterations.m`. The layout of the function is the following:

```

function [xvect, nit] = fixed_point_iterations( phi, x0, tol, nmax )
% FIXED_POINT_ITERATIONS Finds a fixed point of a scalar function.
% [XVECT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) finds a fixed point of
% the iteration function PHI using the fixed point iterations method and
% returns a vector XVECT containing the successive approximations of the
% fixed point (iterates).
% PHI accepts a real scalar input x and returns a real scalar value;
% PHI can also be an inline object. X0 is the initial guess.
% TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
% The stopping criterion based on the difference of successive iterates is used.
% If the search fails a warning message is displayed.
%
% [XVECT,NIT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) also returns the
% number of iterations NIT.
% Note: the length of the vectors is equal to ( NIT + 1 ).
%
return

```

By using the function `fixed_point_iterations`, verify that  $\alpha$  is a fixed point of  $\phi(x)$  for a few choices of  $x^{(0)} \in [a, b]$ , with  $tol = 10^{-6}$  and  $k_{max} = 1500$ .

- b) Report the number of iterations  $k_c$  required for the convergence of the algorithm and the error  $e^{(k_c)} = |x^{(k_c)} - \alpha|$ .
- c) For the convergent cases, plot  $e^{(k)} = |x^{(k)} - \alpha|$  vs.  $k$ . What can be said about the order of convergence of the algorithm?

## Exercise II (Theoretical and MATLAB)

We consider the equation  $f(x) = 0$  in  $[a, b]$ , with  $f(x) = x/2 - \sin(x) + \pi/6 - \sqrt{3}/2$ ,  $a = -\pi/2$  and  $b = \pi$ . The function  $f(x)$  has two zeros,  $\alpha_1 \in I_1 = [-\pi/2, 0]$  and  $\alpha_2 \in I_2 = [\pi/2, \pi]$ . In particular,  $\alpha_1 \simeq -1.047197598567\dots$  and  $\alpha_2 \simeq 2.246005589297\dots$ . We use fixed point iterations to find the zeros  $\alpha_1$  and  $\alpha_2$ , with the iteration function  $\phi(x) = \sin(x) + x/2 - \pi/6 + \sqrt{3}/2$ .

- a) Verify that  $\alpha_2$  is a fixed point of  $\phi(x)$  and establish if the method is:
  - (a) *globally convergent* in  $I_2$ , i.e. the method converges for all  $x^{(0)} \in I_2$ . (*Hint*: you can use MATLAB to plot and evaluate the function  $\phi(x)$  and its derivative  $\phi'(x)$ .)
  - (b) *locally convergent*, i.e. the method converges to  $\alpha_2$  if  $x^{(0)}$  is sufficiently close to  $\alpha_2$ .
- b) By using the function `fixed_point_iterations`, verify the answer given to point a) for two different choices of  $x^{(0)} \in I_2$ , with  $tol = 10^{-6}$  and  $k_{max} = 1500$ . Report the number of iterations ( $k_c$ ) required for the convergence of the algorithm and the error  $e^{(k_c)} = |x^{(k_c)} - \alpha_2|$ . In addition, for the convergent cases, plot  $e^{(k)} = |x^{(k)} - \alpha_2|$  vs.  $k$ . Moreover, plot the ratio  $a^{(k)} := (x^{(k+1)} - \alpha_2)/(x^{(k)} - \alpha_2)$  vs.  $k$ . Discuss the results obtained.
- c) Repeat point a) for  $\alpha_1 \in I_1$ .
- d) Repeat point b) to verify the convergence to  $\alpha_1 \in I_1$ . Try using  $x^{(0)} = -1.1$  and  $x^{(0)} = -0.9$ .
- e) Consider  $\alpha_2 \in I_2$  and the fixed point algorithm (1). Prove that there exists a positive constant  $0 \leq C < 1$  such that, for all  $x^{(0)} \in I_2$ ,

$$|x^{(k+1)} - \alpha_2| \leq C|x^{(k)} - \alpha_2|, \quad \text{for all } k \geq 0 \quad (2)$$

(*hint*: use the mean value theorem). Compute the value of  $C$  explicitly.

- f) Following point e), prove that, given (2), we can write

$$|x^{(k)} - \alpha_2| \leq C^k |x^{(0)} - \alpha_2|, \quad \text{for all } k \geq 0.$$

Then, assuming that  $\alpha_2$  is unknown, use this result to estimate the minimum number of iterations  $k_{min}$  such that the error  $|x^{(k_{min})} - \alpha_2|$  is smaller than  $tol = 2^{-20}$  for all choices of  $x^{(0)} \in I_2$ .

- g) Is the stopping criterion based on the difference between successive approximations  $x^{(n)}$  satisfactory when using fixed point iterations to find  $\alpha_1$  and  $\alpha_2$ ? Motivate and verify your answer from a theoretical point of view and by using the function `fixed_point_iterations` (set  $tol = 10^{-6}$  and  $k_{max} = 1500$ , and run your code for the cases of the fixed points  $\alpha_1$  and  $\alpha_2$ ). Specifically, set  $x^{(0)} = -1.1$  for finding  $\alpha_1$  and  $x^{(0)} = \pi/2$  for finding  $\alpha_2$ .

### Exercise III (Theoretical)

Consider the function  $f(x) = e^x + 3\sqrt{x} - 2$ , where  $x \in I = [0.02, 0.2]$ .

- Show that  $f$  has a zero  $\alpha \in I$  and that it is unique.
- Consider the fixed point iterations algorithm with the following two iteration functions:

$$\phi_1(x) = \log(2 - 3\sqrt{x}) \quad \text{and} \quad \phi_2(x) = \frac{(2 - e^x)^2}{9}.$$

Which iteration function would you use to find  $\alpha \in I$ ? Why? (Hint: plot them in MATLAB.)

- To verify your answer to the previous point, represent qualitatively the behavior of the fixed point iterations for  $\phi_1(x)$  and  $\phi_2(x)$ , starting from  $x^{(0)} = 0.05$ .

### Exercise IV (OPTIONAL, Theoretical)

Let  $\alpha \in (a, b)$  be a zero of the function  $f : [a, b] \rightarrow \mathbb{R}$ , where  $f \in C^q([a, b])$ , with  $q = \max(m, 2)$  for some  $m \geq 1$ . We say that  $\alpha$  has multiplicity  $m \geq 1$  if  $f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0$  and  $f^{(m)}(\alpha) \neq 0$ .

- Write the Newton method as a fixed point iterations algorithm and define the corresponding iteration function  $\phi_N(x)$ .
- Following point a), prove that  $\phi'_N(\alpha) = 1 - 1/m$  for all  $m \geq 1$ , where  $\alpha$  is a zero with multiplicity  $m$ . (Hint: rewrite the function  $f(x)$  as  $f(x) = (x - \alpha)^m g(x)$  in a neighborhood of  $\alpha$ , with  $g(\alpha) \neq 0$ ).
- From points a) and b), deduce the convergence properties of the Newton method, when interpreted as a fixed point iterations algorithm. Specifically, discuss the cases of a zero  $\alpha$  of multiplicity  $m = 1$  and  $m > 1$ .
- The modified Newton method can also be cast as a fixed point iterations algorithm. Similarly to points a) and b), find the corresponding iteration function  $\phi_{N_m}(\alpha)$  and its derivative  $\phi'_{N_m}(\alpha)$  at a zero  $\alpha$  of multiplicity  $m \geq 1$ . As in point c), discuss the convergence properties of the modified Newton method.
- Interpreting the Newton method as a fixed point iterations algorithm, discuss the quality of the stopping criterion based on the difference of successive approximations  $x^{(n)}$  in the cases of a zero  $\alpha$  of multiplicity  $m = 1$  and  $m > 1$ . What do you expect for  $m = 100$ ?