NIDS Session 5

1) For partitioned RK methods; the order conditions are

order $p=1$ $\sum b_i = 1$ and $\sum \hat{b}_i = 1$

order $p=2$ $\sum b_i \hat{a}_{ij} = \frac{1}{2}$ and $\sum \hat{b}_i a_{ij} = 1/2$ $(\star)$

$\sum b_i a_{ij} = \frac{1}{2}$ and $\sum \hat{b}_i \hat{a}_{ij} = 1/2$

Symplectic Euler

$$\begin{array}{c|c} & 1 \\ \hline & 1 \end{array} + \begin{array}{c|c} & 0 \\ \hline & 1 \end{array}$$

order 1 : clear ✓

$b_1 = 1$, $\hat{b}_1 = 1$

Störmer - Verlet

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} + \begin{array}{c|cc} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array} \begin{array}{l} \leftarrow a_{1j} \\ \leftarrow a_{2j} \\ \\ \end{array}$$

order 1 : $\frac{1}{2} + \frac{1}{2} = 1$ ✓

order 2 : Check $(\star)$

$\sum b_i \hat{a}_{ij} = b_1 \left(\frac{1}{2} + 0\right) + b_2 \left(\frac{1}{2} + 0\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ ✓

$\sum \hat{b}_i a_{ij} = \frac{1}{2}$ ✓

Not of order 3 because Lobatto IIIA and IIIB have order $2s-2$ (i.e. 2 here)

2) i)

(A) $\begin{cases} \dot{u}(t) = u(t)\,(v(t) - 2) & u(t_0) = u_0 \\ \dot{v}(t) = v(t)\,(1 - u(t)) & v(t_0) = v_0 \end{cases}$

Let $I : \mathbb{R}^2 \to \mathbb{R}$

$$I(u, v) = u + v - \ln(u) - 2\ln(v)$$

$$\frac{d}{dt} I(u,v) = \nabla I(u,v) \cdot \begin{pmatrix} \dot u(t) \\ \dot v(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{1}{u} \\ 1 - \frac{2}{v} \end{pmatrix} \cdot \begin{pmatrix} u(v-2) \\ v(1-u) \end{pmatrix}$$

$$= u(v-2) - (v-2) + v(1-u) - 2(1-u)$$

$$= 0 \quad \checkmark$$

(ii) a) $u_0 = 0$

$$\begin{pmatrix} \dot u(t_0) = 0 \\ \dot v(t_0) = v_0 \end{pmatrix}$$

Let's consider $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ v_0 e^{t - t_0} \end{pmatrix}$ (2)

Claim : (2) satisfies (1)

Indeed $\dot{u}(t) = 0 = u(t)(v(t) - 2)$

$\dot{v}(t) = v_0 e^{t - t_0}$

$= v(t)$

Thus, since f is continuously differentiable, (thus locally Lipschitz), the unique maximal solution is $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ v_0 e^{t-t_0} \end{pmatrix} \quad t \geq t_0$

defined for $t \in (t_0, +\infty)$

$|f(x) - f(y)| \leq L|x - y| \qquad L > 0$

for some x, y in a neighborhood

b) Let's define $\begin{cases} u^+(t) = u(t) & t \in [\hat{t}, t_0 + T) \\ v^+(t) = v(t) \end{cases}$

u^+ and v^+ satisfy (1) with initial conditions

$u^+(\hat{t}) = u(\hat{t}) = 0$

$v^+(\hat{t}) = v(\hat{t})$

From question a) u^+ will remain zero on $[\hat{t}, t_0 + T)$

Thus $u^+(t) = u(t) = 0$ on $[\hat{t}, t_0 + T)$

Now define
$$\begin{cases} u^-(t) = u(\hat{t} - t) \\ v^-(t) = v(\hat{t} - t) \end{cases}$$

$$\dot{u}^-(t) = -\dot{u}(\hat{t} - t) = -u(\hat{t}-t)(v(\hat{t}-t) - 2)$$
$$= -u^-(t)(v^-(t) - 2)$$

$$\dot{v}^-(t) = -\dot{v}(\hat{t}-t) = -v(\hat{t}-t)(1 - u(\hat{t}-t))$$
$$= -v^-(t)(1 - u^-(t))$$

$$\begin{cases} \dot{u}^-(t) = -u^-(t)(v^-(t) - 2) \\ \dot{v}^-(t) = -v^-(t)(1 - u^-(t)) \end{cases} \qquad t \in (0, \hat{t} - t_0)$$

$$u^-(0) = u(\hat{t}) = 0$$
$$v^-(0) = v(\hat{t})$$

Analogously to a) $u^-(t) = 0$

Recall $\tilde{v}(t) = v(\hat{t} - t) = 0 \quad t \in [0, \hat{t} - t_0]$

$\Leftrightarrow \quad v(s) = 0 \quad s \in [t_0, \hat{t}]$

$(s = \hat{t} - t)$

Conclusion : $v(t) = 0 \quad t \in [t_0, t_0 + T)$

c) Let $v_0 > 0$

and assume $v(t) \leq 0$

At t_0, $v(t_0) = v_0 > 0$

$\hat{t} \in [t_0, t_0 + T)$ such that $v(\hat{t}) \leq 0$

Since v is continuous, by the intermediate value theorem $\exists \tilde{t} \in [t_0, \hat{t})$ such that $v(\tilde{t}) = 0$

Question b) implies that $v(t) = 0$

$t \in [t_0, t_0 + T)$

But $v(t_0) = v_0 > 0$, we have a contradiction!

Conclusion : $v(t)$ remains positive

3) We consider the system

$$\begin{cases} \dot{Y}(t) = A(Y(t))\, Y(t) \\ Y(t_0) = Y_0 \end{cases} \qquad \begin{array}{l} Y(t) \in \mathbb{R}^{m \times m} \\ A(Y) \in \mathbb{R}^{m \times m} \end{array}$$

i) We want to show that

$I(Y) = Y(t)^T Y(t)$ is a first integral

$$\frac{d}{dt}\left(I(Y)\right) = \dot{Y}^T(t) Y(t) + Y(t)^T \dot{Y}(t)$$

$$= Y(t)^T A(Y)^T Y(t) + Y(t) A(Y) Y(t)$$

$$= Y(t)^T \big(\underbrace{A(Y)^T}_{= -A(Y)} + A(Y)\big) Y(t)$$

$$= 0$$

ii) The Gauss methods conserve quadratic first integral, thus conserve $I(Y)$

iii) If $Y_0^T Y_0 = I$

$$I(Y) = Y(t)^T Y(t) = I(Y_0) = Y_0^T Y_0 = I$$

$$\Rightarrow \quad Y(t)^T Y(t) = I$$

$Y(t)$ remains orthogonal

$\boxed{c}$ We consider

$$(1)\begin{cases} \dot{y}_1(t) = \alpha_1 \, y_2(t) \, y_3(t) & \alpha_1 = \frac{1}{I_3} - \frac{1}{I_2} \\ \dot{y}_2(t) = \alpha_2 \, y_1(t) \, y_3(t) & \alpha_2 = \frac{1}{I_1} - \frac{1}{I_3} \\ \dot{y}_3(t) = \alpha_3 \, y_1(t) \, y_2(t) & \alpha_3 = \frac{1}{I_2} - \frac{1}{I_1} \end{cases}$$

Idea: Rewrite (1) as $\quad \dot{y}(t) = A(y) \, y(t)$

$$\dot{y}_1(t) = \left(\frac{1}{I_3} - \frac{1}{I_2} \right) y_2 y_3$$

$$= \left(0 \quad \frac{1}{I_3} y_3(t) \quad -\frac{1}{I_2} y_2(t) \right) \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

Analogously for the other equations:

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{y_3(t)}{I_3} & -\frac{y_2(t)}{I_2} \\ -\frac{y_3}{I_3} & 0 & \frac{y_1}{I_1} \\ \frac{y_2}{I_2} & -\frac{y_1}{I_1} & 0 \end{pmatrix}}_{A(y)} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

A is skew symmetric if $A^T = -A$

$(\Rightarrow a_{ii} = 0)$

Note that $A(y)$ is skew symmetric!

From exercise 3, we know that

$y(t)^T y(t)$ is a first integral, which here

reduces to $y(t) \cdot y(t) = \|y(t)\|_2^2$ (1)

ii) Again, Gauss methods in particular conserve (1)

For instance, for $s = 1$

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

(Implicit midpoint rule)

$s = 2$

$$\begin{array}{c|cc} \frac{3-\sqrt{3}}{6} & 1/4 & 1/4 - \sqrt{3}/6 \\ \frac{3+\sqrt{3}}{6} & 1/4 + \sqrt{3}/6 & 1/4 \\ \hline & 1/2 & 1/2 \end{array}$$

iii) We want to show that

$$H(y) = \frac{1}{2}\left(\frac{y_1^2}{I_1} + \frac{y_2^2}{I_2} + \frac{y_3^2}{I_3}\right)$$

Let compute

$$\frac{d}{dt}\left(H(y)\right) = \nabla H(y(t)) \cdot \dot{y}(t)$$

$$= \begin{pmatrix} \frac{y_1}{I_1} \\ \frac{y_2}{I_2} \\ \frac{y_3}{I_3} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 y_2 y_3 \\ \alpha_2 y_1 y_3 \\ \alpha_3 y_2 y_3 \end{pmatrix}$$

$$= \left(\frac{\alpha_1}{I_1} + \frac{\alpha_2}{I_2} + \frac{\alpha_3}{I_3}\right) y_1 y_2 y_3$$

$$= \left(\frac{1}{I_1 I_3} - \frac{1}{I_1 I_2} + \frac{1}{I_2 I_1} - \frac{1}{I_2 I_3} + \frac{1}{I_3 I_2} - \frac{1}{I_3 I_1}\right) y_1 y_2 y_3$$

$$= 0$$

Thus, $H(y)$ is another first integral.

5 We know from exercise 4 that both $I(q)$ and $H(q)$ are first integral. Hence

$$\begin{cases} I(q) = I(q_0) & \rightarrow \text{ defines a sphere} \\ H(q) = H(q_0) & \rightarrow \text{ defines an ellipsoid} \end{cases}$$