NIDS Session 1

$\boxed{1}$ $p, q : \mathbb{R}^+ \to \mathbb{R}^2$

$$\begin{cases} \dot{p}(t) = -\dfrac{\partial H}{\partial q}(p(t), q(t)) & (1) \\ \dot{q}(t) = \dfrac{\partial H}{\partial p}(p(t), q(t)) & (2) \end{cases}$$

$$H(p(t), q(t)) = \frac{1}{2} \|p(t)\|_2 - \frac{1}{\|q(t)\|_2}$$

$$\begin{cases} \dot{p}(t) = -\dfrac{q(t)}{\|q(t)\|^3} \\ \dot{q}(t) = p(t) \end{cases}$$

- Explicit Euler:

$$\begin{cases} p_{n+1} = p_n - \underline{h \dfrac{\partial H}{\partial q}(p_n, q_n)} \\ q_{n+1} = q_n + \underline{h \dfrac{\partial H}{\partial p}(p_n, q_n)} \end{cases}$$

explicit method

Motivation: From (1) at time t_n

$$\dot{p}(t_n) = -\frac{\partial H}{\partial q}(p(t_n), q(t_n))$$

$$\dot{p}(t_n) \approx \frac{p(t_{n+1}) - p(t_n)}{h}$$

h: step size

$$p_n \approx p(t_n)$$

$$\frac{p_{n+1} - p_n}{h} = -\frac{\partial H}{\partial q}(p_n, q_n)$$

$$p_{n+1} = p_n - h \frac{\partial H}{\partial q}(p_n, q_n)$$

- Symplectic Euler

$$p_{n+1} = p_n - h \frac{\partial H}{\partial q}(p_n, q_{n+1}) \qquad \text{Explicit} \qquad (3)$$

$$\underline{q_{n+1}} = q_n + h \frac{\partial H}{\partial p}(p_n, \underline{q_{n+1}}) \qquad \underline{\text{Implicit}} \qquad (4)$$

First solve (4) $\rightarrow q_{n+1}$

then solve (3) $\rightarrow p_{n+1}$

. Implicit Euler

$$p_{n+1} = p_n - h \frac{\partial H}{\partial q}(p_{n+1}, q_{n+1})$$ implicit

$$q_{n+1} = q_n + h \frac{\partial H}{\partial p}(p_{n+1}, q_{n+1})$$ implicit

method	Error in h	Error in L	global error
Explicit Euler	$O(ht)$	$O(ht)$	$O(ht^2)$
Symplectic Euler	$O(h)$	0	$O(th)$
Implicit Euler	$O(ht)$	$O(ht)$	$O(ht^2)$
midpoint rule	$O(h^2)$	0	$O(h^2t)$

← trust me!

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$$L(p,q) = q_1p_2 - q_2p_1$$

$$\frac{d}{dt}\left(L(p(t), q(t)) \right) = 0 \Rightarrow L(p(t), q(t)) = L(p(0), q(0))$$

$$\frac{d}{dt}\left(L(p(t), q(t)) \right) = \frac{\partial L}{\partial p} \cdot \dot{p}(t) + \frac{\partial L}{\partial q} \cdot \dot{q}(t)$$

$$= \begin{pmatrix} -q_2(t) \\ q_1(t) \end{pmatrix} \cdot \begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix} + \begin{pmatrix} p_2(t) \\ -p_1(t) \end{pmatrix} \cdot \begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix}$$

$$= -\frac{1}{\|q(t)\|^3} \underbrace{\begin{pmatrix} -q_2(t) \\ q_1(t) \end{pmatrix} \cdot \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}}_{=0} + \underbrace{\begin{pmatrix} p_2(t) \\ -p_1(t) \end{pmatrix} \cdot \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}}_{=0}$$

$$= 0$$

$$\boxed{\dot{y}(t) = f(y(t))}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$\left.\begin{aligned} \dot{p}(t) &= -\frac{\partial H}{\partial q}(p(t), q(t)) \\ \dot{q}(t) &= \frac{\partial H}{\partial p}(p(t), q(t)) \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}_{J^{-1}} \begin{pmatrix} \frac{\partial H}{\partial p}(p(t), q(t)) \\ \frac{\partial H}{\partial q}(p(t), q(t)) \end{pmatrix}$$

$$\dot{y}(t) = J^{-1} \nabla H(y)$$

$$J^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}}_{J} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Show that $| H(y_n) - H(y_0) | \leq C t h^p$

H: Hamiltonian is assumed Lipschitz continuous

$$| H(x) - H(y) | \leq L \, | x - y | \quad (x, y \in \mathbb{R}^d) \quad L > 0$$

Note that:

$$| H(y_n) - H(y_{n-1}) + H(y_{n-1}) + \dots + -H(y_1) + H(y_1) - H(y_0) |$$

$$= \left| \sum_{k=0}^{n-1} H(y_{k+1}) - H(y_k) \right|$$

$$\leq \sum_{k=0}^{n-1} | H(y_{k+1}) - H(y_k) |$$

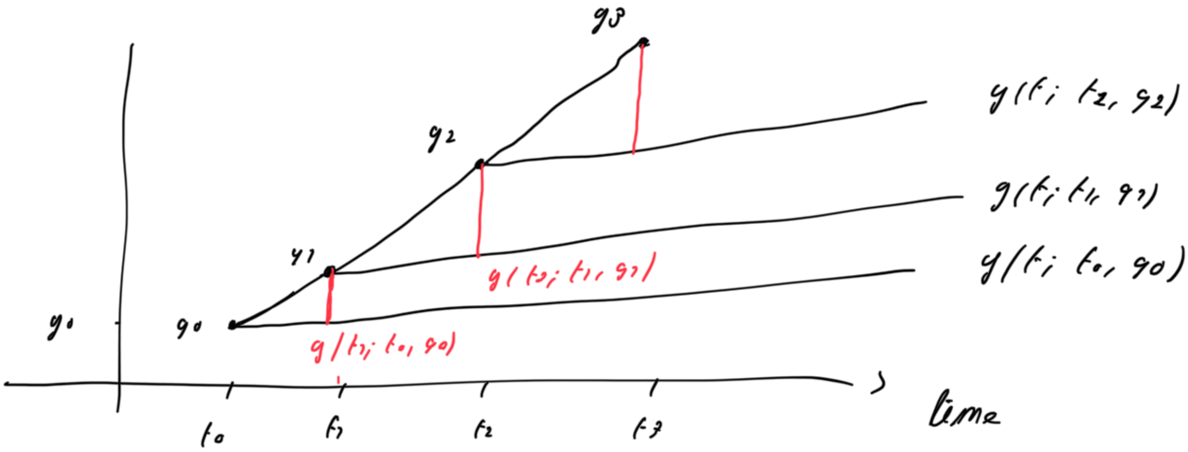
$$\begin{cases} g'(t) = f(g(t)) & (1) \\ g(t_0) = g_0 \end{cases}$$

The unique solution of (1) is denoted $g(t; \underbrace{t_0, g_0}_{\text{parameters}})$

time of the solution

time for initial condition

initial condition



If the RK has order p

$$|g_1 - g(t_1, t_0, g_0)| \le c\, h^{p+1} \qquad (5)$$

$$|g_{k+1} - g(t_{k+1}, t_k, g_k)| \le c\, h^{p+1}$$

Define the exact flow $\varphi_t(g_0) = g(t; 0, g_0)$

$$|g_1 - \varphi_h(g_0)| \le c\, h^{p+1} \quad \text{(rewriting (5))}$$

We had

$$|H(g_n) - H(g_0)| \le \sum_{k=0}^{n-1} |H(g_{k+1}) - H(g_k)|$$

H is a first integral $\Rightarrow$ $H(g_k) = H(\varphi_h(g_k))$

$$|H(g_n) - H(g_0)| \le \sum_{k=0}^{n-1} |H(g_{k+1}) - H(\varphi_h(g_k))|$$

$$\leq L \sum_{k=0}^{n-1} | y_{k+1} - \varphi_h(y_k) |$$

$$\leq Lc \sum_{k=0}^{n-1} h^{p+1} \qquad \text{(Method of order } p\text{)}$$

$$= cLnh^{p+1} = cL\underbrace{nh}_{t}h^{p} = cLth^{p}$$