NIAS Session 13

$\boxed{1}$ Recall that the stability function of a RK method is

$$R(\beta) = 1 + \beta b^T (I - \beta A)^{-1} \mathbb{1} \quad \text{where } \beta = h\lambda$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{pmatrix} \quad \mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Recall the Neumann series expansion, for $B \in M_n(\mathbb{R})$

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \quad \text{provided that } \|B\| < 1$$

Applying this result to $B = \beta A$, we obtain

$$R(\beta) = 1 + \beta b^T (I + \beta A + O(\beta^2)) \mathbb{1}$$

provided that $\|\beta A\| < 1 \quad \left(h < \frac{1}{|\lambda| \, \|A\|} \right)$

$$R(\beta) = 1 + \beta b^T \mathbb{1} + \beta^2 b^T A \mathbb{1} + O(\beta^3). \quad \text{Thus}$$

$$R(\beta) = 1 + \beta + \frac{1}{2}\beta^2 + O(\beta^3)$$

$$\Leftrightarrow \underbrace{\begin{cases} b^T \mathbb{1} = 1 \rightarrow \sum_{i=1}^{n} b_i = 1 & \text{order 1} \\ b^T A \mathbb{1} = \frac{1}{2} \rightarrow \sum_{i,j=1}^{n} b_i a_{ij} = \frac{1}{2} & \text{order 2} \end{cases}}_{\text{order conditions}}$$

$\boxed{2}$ Let $n \in \mathbb{N}$ $x \in [-1, 1]$, define

$$T_n(x) = \cos(n \arccos(x))$$

i) Denoting $\varphi = \arccos(x)$

for $n = 0$ $T_0(x) = \cos(0) = 1$

$n = 1$ $T_1(x) = \cos(\varphi) = x$

Recall that $2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$2\cos(\varphi)\cos((n-1)\varphi) = \cos(((n-1)+1)\varphi) + \cos((n-1)-1)\varphi)$$

$$= \cos(n\varphi) + \cos((n-2)\varphi)$$

By definition, $2x\, T_{n-1}(x) = T_n(x) + T_{n-2}(x)$

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

ii) $T_n\left(\cos\left(\frac{k\pi}{n}\right)\right) = \cos(k\pi) = (-1)^k$ $k = 0, .., n$

$$= \cos\left(n \arccos\left(\cos\left(\frac{k\pi}{n}\right)\right)\right)$$

$$T_n\left(\cos\left(\frac{(2k+1)\pi}{2n}\right)\right) = \cos\left(\frac{(2k+1)\pi}{2}\right) = 0$$

$k = 0, .., n-1$

iii) $\displaystyle\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx$ (1) $\quad x = \cos(t)$

$\quad dx = -\sin(t)\, dt$

$$\sqrt{1-\cos^2(t)} = \sin(t)$$

(1) Becomes

$$-\int_{\pi}^{0} \cos(nt)\cos(mt)\, dt = \int_{0}^{\pi} \cos(nt)\cos(mt)\, dt$$

$$= \frac{1}{2}\int_{0}^{\pi} \cos((n+m)t) + \cos((n-m)t)\, dt \qquad (2)$$

- If $n \neq m \neq 0$, then (2) becomes

$$\frac{1}{2}\left(\frac{1}{n+m}\sin((n+m)t) + \frac{1}{n-m}\sin((n-m)t)\right)\Big|_{0}^{\pi} = 0$$

- If $n = m \neq 0$, then (2) reduces to

$$\frac{1}{2}\int_{0}^{\pi} (\cos(2nt) + 1)\, dt = \frac{1}{2}\left(\frac{1}{2n}\sin(2nt) + t\right)\Big|_{0}^{\pi}$$

$$= \frac{\pi}{2}$$

- If $n = s = 0$, then (1) becomes

$$\int_0^\pi 1 \, dt = \pi$$

Conclusion :

$$\int_{-1}^{1} \frac{T_n(x)\, T_k(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} 0 & \text{if } n \neq k \\ \frac{\pi}{2} & \text{if } n = k \neq 0 \\ \pi & \text{if } n = k = 0 \end{cases}$$

iv) We prove by induction that

$$T_n'(1) = n^2 \quad \text{and} \quad T_n''(1) = \frac{1}{3} n^2 (n^2 - 1) \qquad (2)$$

Initialization :

- $n = 0$ $\quad T_0(x) = 1 \Rightarrow T_0'(x) = 0 = T_0''(x)$

$$= n^2 \big|_{n=0} \qquad = \frac{1}{3} n^2 (n^2-1) \big|_{n=0}$$

- $n = 1$ $\quad T_1(x) = x \quad T_1'(x) = 1 \quad T_1''(x) = 0$

$$= n^2 \big|_{n=1} \qquad = \frac{1}{3} n^2 (n^2-1) \big|_{n=1}$$

Induction : Assume (1) and (2) hold for all $k \leq n-1$

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

$$T_n'(x) = 2\,T_{n-1}(x) + 2x\,T_{n-1}'(x) - T_{n-2}'(x)$$

$$T_n''(x) = 4\,T_{n-1}'(x) + 2x\,T_{n-1}''(x) - T_{n-2}''(x)$$

Prove that $T_n(1) = 1 \quad \forall n$

Thus

$$T_n'(1) = 2\,\overbrace{T_{n-1}(1)} + 2\,T_{n-1}'(1) - T_{n-2}'(1)$$

$$= 2 + 2(n-1)^2 - (n-2)^2$$

$$= n^2$$

$$T_n''(1) = 4(n-1)^2 + \frac{2}{3}(n-1)^2\big((n-1)^2 - 1\big)$$

$$- \frac{1}{3}(n-2)^2\big((n-2)^2 - 1\big)$$

$$= \frac{1}{3} n^2 (n^2 - 1)$$

$\boxed{3}$ Consider the RK Chebyshev method

$$\begin{cases} g_0 = y^0 \\ g_1 = g_0 + \dfrac{\Delta t}{n^2} f(g_0) \\ g_i = \dfrac{2\Delta t}{n^2} f(g_{i-1}) + 2 g_{i-1} - g_{i-2} \qquad i = 2, \dots, n \\ y^1 = g_n \end{cases}$$

i) We show that $g_i = T_i\left(1 + \frac{\Delta t \lambda}{s^2}\right) g_0$

To simplify the notation, let $\chi = 1 + \frac{\Delta t \lambda}{s^2}$

Initialisation:

$$g_0 = g_0 = \underbrace{T_0(\chi)}_{1} g_0$$

$$g_1 = \left(1 + \frac{\Delta t \lambda}{s^2}\right) g_0 = T_1(\chi) g_0$$

Induction:

$$g_i = \frac{2\Delta t}{s^2} f(g_{i-1}) + 2 g_{i-1} - g_{i-2}$$

$$= 2 \underbrace{\left(1 + \frac{\Delta t \lambda}{s^2}\right)}_{=\chi} g_{i-1} - g_{i-2}$$

$$= \left(2 \chi T_{i-1}(\chi) - T_{i-2}(\chi)\right) g_0$$

$$= T_i(\chi) g_0$$

In particular $g_1 = g_s = T_s(\chi) g_0$

ii) We prove that $g_i = g_0 + \Delta t \sum_{j=0}^{i-1} a_{ij} f(g_j)$ (1)

$i = 1 .., s$

where $k_i = f(g_i)$

Note that if (1) holds

$$f(g_i) = k_i = f\left(g_0 + \Delta t \sum_{j=0}^{i-1} a_{ij} k_j \right)$$

is in the RK framework.

Let's prove (1):

Set $a_{0k} = 0 \quad \forall k$

Initialisation at $i = 1$

$$g_1 = g_0 + \frac{\Delta t}{s^2} f(g_0) = g_0 + \Delta t \, a_{10} f(g_0)$$

with $a_{10} = \frac{1}{s^2}$

Induction: Assume (1) holds for all $i \leq k-1$

$$g_k = \frac{2\Delta t}{s^2} f(g_{k-1}) + 2 g_{k-1} - g_{k-2}$$

$$= \frac{2\Delta t}{s^2} f(g_{k-1}) + 2\left(g_0 + \Delta t \sum_{j=0}^{k-2} a_{k-1,j} f(g_j) \right) - \left(g_0 + \Delta t \sum_{j=0}^{k-3} a_{k-2,j} f(g_j) \right)$$

$$= g_0 + \Delta t \sum_{j=0}^{k-1} a_{kj} f(g_j)$$

with coefficients :

$$\begin{cases} a_{k,k-1} = \dfrac{2}{s^2} \\ a_{k,k-2} = 2\, a_{k-1,k-2} \\ a_{k,j} = 2\, a_{k-1,j} - a_{k-2,j} \qquad j = 0, \dots, k-3 \end{cases}$$

(iii) We prove by induction that $\displaystyle\sum_{j=j}^{i-1} a_{ij} = \left(\frac{i}{s}\right)^2$

Initilisation for $i = 1$

$$\sum_{j=0}^{0} a_{1j} = a_{10} = \frac{1}{s^2} = \left(\frac{i}{s}\right)^2 \Big|_{i=1}$$

Induction :

$$\sum_{j=0}^{k-1} a_{kj} = \sum_{j=0}^{k-3} a_{kj} + a_{k,k-2} + a_{k,k-1}$$

$$= \sum_{j=0}^{k-3} 2\, a_{k-1,j} - a_{k-2,j} + 2\, a_{k-1,k-2} + \frac{2}{s^2}$$

$$= 2 \sum_{j=0}^{k-2} a_{k-1,j} - \sum_{j=0}^{k-3} a_{k-2,j} + \frac{2}{s^2}$$

$$= 2 \left(\frac{k-1}{s}\right)^2 - \left(\frac{k-2}{s}\right)^2 + \frac{2}{s^2}$$

$$= \left(\frac{k}{s}\right)^2$$

iv) $y_1 = y_s = y_0 + \Delta t \sum_{j=0}^{s-1} a_{sj} f(y_j)$

$$= y_0 + \Delta t \sum_{j=0}^{s-1} b_j k_j$$

where $b_j = a_{sj}$ $\quad j = 0, \dots, s-1$

Note that $\sum_{j=0}^{s-1} b_j = \sum_{j=0}^{s-1} a_{sj} = \left(\frac{s}{s}\right)^2 = 1$

order 1 condition satisfied

v) Consider the error $y_1 - y(\Delta t)$ for the function $f(y) = \lambda y$, we know that

$$y(\Delta t) = e^{\lambda \Delta t} y_0$$

$$y_1 = R(z)\, y_0 = T_s\left(1 + \frac{\Delta t \lambda}{s^2}\right) y_0$$

Therefore

$$y_1 - y(\Delta t) = \left(T_s\left(1 + \frac{\Delta t \lambda}{s^2}\right) - e^{\lambda \Delta t}\right) y_0$$

$$= \left(T_s(1) + T_s'(1) \frac{\Delta t \lambda}{s^2} + \frac{1}{2} T_s''(1) \left(\frac{\Delta t \lambda}{s^2}\right)^2 + O(\Delta t^3) - e^{\lambda \Delta t} \right) y_0$$

$$= \left(1 + n^2 \frac{\Delta t \lambda}{n^2} + \frac{1}{2}\left(\frac{\Delta t \lambda}{n^2}\right)^2 \frac{2}{3} n^2(n^2-1) + O(\Delta t^3) - e^{\lambda \Delta t}\right) y_0$$

$$= \left(1 + \Delta t \lambda + \frac{1}{6} (\Delta t \lambda)^2 \frac{(n^2-1)}{n^2} + O(\Delta t^3) - e^{\lambda \Delta t}\right) y_0$$

But $\frac{1}{6} \frac{(n^2-1)}{n^2} = \frac{1}{6}\left(1 - \frac{1}{n^2}\right) < \frac{1}{6} < \frac{1}{2}$

Consequently, $y_1 - y(\Delta t) = O(\Delta t^2)$ and the method does not have order 2

vi) Recall from Series 2 that

$$c_i = \sum_{j=0}^{n-1} a_{ij} = \left(\frac{i}{n}\right)^2$$

$\boxed{4}$ i) Recall from exercise 3 that

$$y_1 = T_n\left(1 + \frac{\Delta t \lambda}{n^2}\right) y_0$$

Since $|T_n(x)|$ exceeds 1 when $|x| > 1$, we must require that $-1 \leq 1 + \frac{\Delta t \lambda}{n^2} \leq 1$

$$1 + \frac{\Delta t \lambda}{s^2} \geq 1 \Rightarrow \Delta t \leq \frac{2s^2}{|\lambda|}$$

Hence, increasing s extends the stability region!

ii) If Δt is fixed $\quad s \geq \left\lceil \sqrt{\frac{\Delta t |\lambda|}{2}} \right\rceil$

where $\lceil \cdot \rceil$ is the ceil function

iii) Number of time steps?

Assuming that Δt is the largest possible for each method while remaining stable

. For explicit Euler: $\Delta t_e = \frac{2}{|\lambda|} \Rightarrow N_e = \frac{T}{\Delta t_e} = \frac{T|\lambda|}{2}$

. For RKC method: $\Delta t_c = \frac{2s^2}{|\lambda|} \Rightarrow N_c = \frac{T}{\Delta t_c} = \frac{T|\lambda|}{2s^2}$

steps

Number of function evaluations?

. Explicit Euler $\quad y_1 = y_0 + \Delta t f(y_0)$

$\rightarrow$ 1 function evaluation

• RKC $\quad y_1 = y_s = y_0 + \Delta t \sum_{j=0}^{s-1} a_{sj} f(y_j)$

→ s function evaluations

Conclusion : Total number of evaluations :

• Explicit Euler : $\quad 1 \cdot N_c = \frac{T|\lambda|}{2}$

• RKC : $\quad s \cdot N_c = \frac{T|\lambda|}{2s} \leq \frac{T|\lambda|}{2}$

Conclusion : even though the RKC method requires more function evaluations in each step, it is still cheaper that Explicit Euler due to its larger step size.