Series 9

Exercise 1. Show by a direct computation that the "symplectic Euler method" is symplectic.

Exercise 2. Consider a general Hamiltonian system of the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(p,q), \quad \dot{q}_i = \frac{\partial H}{\partial p_i}(p,q), \qquad i = 1, \dots, d, \tag{1}$$

where $p, q \in \mathbb{R}^d$ and $H\colon \mathbb{R}^{2d} \to \mathbb{R}$ denotes the Hamiltonian function.

The aim of this exercise is to prove the following statement in E. Celledoni, R. McLachlan, D. McLaren, B. Owren, G. Quispel and W. Wright, *Energy-preserving Runge–Kutta methods*, ESAIM: M2AN, vol. 43, 2009:

"No consistent Runge–Kutta method exactly preserves the Hamiltonian for all polynomial Hamiltonian systems. However, for any given polynomial Hamiltonian, there exists a consistent Runge–Kutta method that exactly preserves it."

i) Show that a consistent Runge–Kutta method cannot exactly preserve the Hamiltonian $H(p,q)$ for all Hamiltonian systems (3) with polynomial $H(p,q)$.
Hint: Let $d = 1$ and consider the Hamiltonian $H(p,q) = p - \int_0^q g(t)\,\mathrm{d}t$ where $g(t)$ is some polynomial.

ii) Consider the average vector field (AVF) method

$$y_{n+1} = y_n + h\int_0^1 f(\theta y_{n+1} + (1-\theta)y_n)\,\mathrm{d}\theta,$$

where $y_n \in \mathbb{R}^{2d}$, and assume that the vector field $f\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is L-Lipschitz continuous. Show that this method is well-defined for a step size h small enough.
Hint: use a fixed-point theorem.

iii) Show that the AVF method exactly conserves the energy $H(y)$ for any system (3).
Hint: prove and use that $H(y_{n+1}) - H(y_n) = \displaystyle\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(H(\theta y_{n+1} + (1-\theta)y_n)\Big)\,\mathrm{d}\theta$.

iv) Assume that the Hamiltonian function $H(p,q)$ is polynomial. Show that there exists a quadrature formula $(b_i, c_i)_{i=1,\dots,s}$, with nodes c_i and weights b_i, such that

$$\int_0^1 f(\theta y_{n+1} + (1-\theta)y_n)\,\mathrm{d}\theta = \sum_{i=1}^{s} b_i f(y_n + (y_{n+1} - y_n)c_i),$$

where $f(y) = J^{-1}\nabla H(y)$.

v) Using point *iv*), construct a Runge–Kutta method which coincides with the AVF method for this particular polynomial Hamiltonian $H(p,q)$ and deduce that this Runge–Kutta method exactly preserves the Hamiltonian.

Exercise 3. Find a consistent, irreducible Runge–Kutta method which is symplectic, but different from all Gauss methods.

Exercise 4. Consider the mathematical model for a pendulum

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q), \qquad \dot{q} = \frac{\partial H}{\partial p}(p, q), \tag{2}$$

where $H(p, q) = \frac{1}{2}p^2 - \cos(q)$ and with a set of initial values

$$p_0^i = \sin(\alpha_i)/4, \quad q_0^i = \pi/2 + \cos(\alpha_i)/4, \qquad i = 1, \ldots, 100,$$

where $\alpha_i = 2\pi i/100$.

i) For all $i = 1, \ldots, 100$, solve problem (5) from $t_0 = 0$ to $T = 10$ with (p_0^i, q_0^i) as initial condition and using the implicit midpoint rule.

ii) For different step sizes $h = 0.1, 0.2, \ldots, 2$ verify that the area defined by the polygon (p^i, q^i), $i = 1, \ldots, 100$, is conserved at the final time $T = 10$.
Hint: to compute the area of a polygon in MATLAB use the function `polyarea`.

iii) Repeat points *i)* and *ii)* using the explicit midpoint rule. Is the area defined by the polygon (p^i, q^i), $i = 1, \ldots, 100$, still conserved?