Series 8

Exercise 1. Consider the Hamiltonian system

$$\dot{y} = f(y) = J^{-1}\nabla H(y), \qquad y(0) = y_0, \tag{1}$$

where $f\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, and its variational equation

$$\dot{\Psi} = \frac{\partial f}{\partial y}\Psi, \qquad \Psi(0) = I_{2n}. \tag{2}$$

i) Show that $\operatorname{tr}\left(\frac{\partial f}{\partial y}\right) = 0$ and deduce that $\det(\Psi)$ is a first integral of (2).
Hint: use the Abel–Liouville–Jacobi formula which states that for a system of differential equations $\dot{\Phi}(t) = A(t)\Phi(t)$ it holds $\frac{\mathrm{d}}{\mathrm{d}t}\det(\Phi(t)) = \operatorname{tr}(A(t))\det(\Phi(t))$.

ii) Prove the Liouville's theorem. In particular, show that for every bounded open set $\Omega \subset \mathbb{R}^{2n}$ and for every $t \in \mathbb{R}$, for which the flow φ_t of (1) exists, it holds

$$\operatorname{Vol}(\varphi_t(\Omega)) = \operatorname{Vol}(\Omega),$$

where $\operatorname{Vol}(\Omega) = \int_\Omega \mathrm{d}y$, which means that the flow is volume preserving.

Consider now a general system of differential equations $\dot{y} = f(y)$ in $\mathbb{R}^d$.

iii) Show that its flow φ_t is volume preserving if and only if $\operatorname{div}(f(y)) = 0$ for all $y \in \mathbb{R}^d$.

Exercise 2.

i) Show that a smooth transformation $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic if and only if it is volume and orientation preserving.
Hint: a map g is said to be orientation preserving if $\det(g'(y)) > 0$ for all $y \in \mathbb{R}^2$.

ii) Is the statement in point *i)* still true for a transformation $g\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $n > 1$? If yes, prove it, and if not, find a counterexample.

Exercise 3. Let $\psi\colon U \to V$, with $U, V \subset \mathbb{R}^d$ be a change of coordinates such that ψ and ψ^{-1} are continuously diferentiable. Prove that if ψ is symplectic, then the Hamiltonian system (1) reads in the new variable $z(t) = \psi(y(t))$

$$\dot{z} = J^{-1}\nabla K(z), \qquad z(0) = z_0, \tag{3}$$

where $K(z) = H(\psi^{-1}(z))$, i.e., $K(z) = H(y)$. Moreover, show that if ψ transforms every Hamiltonian system of the form of (1) to another Hamiltonian system via (4), then ψ is symplectic.