Series 7

Exercise 1. Let $\rho\colon \mathbb{R}^d \to \mathbb{R}^d$ be a linear invertible map and $y' = f(y)$ be a ρ-reversible problem, i.e., $\rho \circ f = -f \circ \rho$. Show that the numerical flow $\phi_h\colon \mathbb{R}^d \to \mathbb{R}^d$, $y_0 \mapsto y_1$, of any Runge–Kutta method satisfies the property $\rho \circ \phi_h = \phi_{-h} \circ \rho$.

Exercise 2. Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz continuous and consider the differential equation

$$y' = f(y), \qquad y(0) = y_0.$$

Assume that a unique solution exists for all $t \in \mathbb{R}$ and define the flow of the differential equation

$$\varphi_t(y_0) \coloneqq y(t; 0, y_0),$$

which denotes the solution at time t with initial value y_0 at time 0.

i) Show that $\varphi_t \circ \varphi_s(y_0) = \varphi_{t+s}(y_0)$ and deduce that $\varphi_t \circ \varphi_{-t} = \mathrm{Id}$ for all $t, s \in \mathbb{R}$.
Hint: use an argument similar to Exercise 2 *ii*) *b*) of Series 5.

Given a numerical method $y_{n+1} = \Phi_h(y_n)$, its adjoint $y_{n+1} = \Phi_h^*(y_n)$ is the method defined as $y_{n+1} = \Phi_{-h}^{-1}(y_n)$ or equivalently $y_n = \Phi_{-h}(y_{n+1})$. Moreover, we say that a numerical method $y_{n+1} = \Phi_h(y_n)$ is symmetric if it satisfies $\Phi_h \circ \Phi_{-h} = \mathrm{Id}$. Show that:

ii) Φ_h symmetric $\Leftrightarrow \Phi_h = \Phi_h^*$,
iii) $(\Phi_h^*)^* = \Phi_h$,
iv) $(\Phi_h \circ \Psi_h)^* = \Psi_h^* \circ \Phi_h^*$ for all one-step methods Φ_h and Ψ_h.

Exercise 3. Consider an s-stage Runge–Kutta method which is consistent, i.e., $\sum_{i=1}^s b_i = 1$, and whose coefficients are such that $\sum_{j=1}^s a_{ij} = c_i$ for all $i = 1, \dots, s$.

i) Show that the adjoint of the Runge–Kutta method is still a Runge–Kutta method with coefficients

$$a_{ij}^* = b_{s+1-j} - a_{s+1-i,s+1-j}, \quad b_i^* = b_{s+1-i}, \quad c_i^* = 1 - c_{s+1-i}, \qquad i, j = 1, \dots s. \tag{1}$$

Hint: use the convention that the c_i and the c_i^* are ordered, i.e., $c_1 \le \cdots \le c_s$ and $c_1^* \le \cdots \le c_s^*$.
ii) Deduce that if $a_{ij} = b_j - a_{s+1-i,s+1-j}$ for all $i, j = 1, \dots, s$ then the method is symmetric.
iii) Deduce that if the Runge–Kutta method is explicit then it cannot be symmetric.

Exercise 4. Show that if an irreducible Runge–Kutta method conserves quadratic invariants then so does its adjoint.