Series 5

Exercise 1. Show that the symplectic Euler method has order $p = 1$ and the Störmer–Verlet method has order $p = 2$.

Exercise 2. (Lotka–Volterra) Let us consider the Lotka–Volterra system

$$\begin{cases} \dot{u} = u(v-2), & u(t_0) = u_0, \\ \dot{v} = v(1-u), & v(t_0) = v_0. \end{cases} \tag{1}$$

i) Show that the following quantity is a first integral

$$I(u, v) = u + v - \ln u - 2 \ln v.$$

ii) Let T denote the length of the interval of existence of the solution of (1).

a) Show that if $u_0 = 0$ then $T = +\infty$ and $u(t) = 0$ for all $t \geq t_0$.

b) Show that if there exists $\tilde{t} \in [t_0, t_0 + T[$ such that $u(\tilde{t}) = 0$ then $u(t) = 0$ for all $t \in [t_0, t_0 + T[$.

c) Show that if $u_0 > 0$ then $u(t) > 0$ for all $t \in [t_0, t_0 + T[$.

Hint: use the Cauchy–Lipschitz theorem with the result about maximality of solutions, which states that if a vector field f is locally Lipschitz continuous, then for all initial values y_0 the system

$$\dot{y} = f(y), \qquad y(t_0) = y_0,$$

has a unique maximal solution $y(t)$ on an interval of the form $[t_0, t_0 + T_{y_0}[$ with length satisfying $0 < T_{y_0} \leq +\infty$.

iii) Implement the Gauss methods for $s = 1, 2$ and verify that the first integral $I(u, v)$ is not conserved. Set the initial conditions $u_0 = v_0 = 2$, the initial time $t_0 = 0$, the final time $T = 30$ and the step size $h = 0.1$.

Exercise 3. We consider the system

$$\dot{Y} = A(Y)Y, \quad Y(t_0) = Y_0,$$

where $Y \in \mathbb{R}^{n \times m}$ and $A(Y) \in \mathbb{R}^{n \times n}$.

i) Show that if $A(Y)$ is a skew-symmetric matrix for all Y, i.e., $A(Y)^\top = -A(Y)$, then $I(Y) = Y^\top Y$ is a first integral of the system.

ii) Which Runge–Kutta methods conserve $I(Y)$?

iii) Show that if in addition $m = n$ and Y_0 is orthogonal, i.e., $Y_0^\top Y_0 = I_n$, then the solution $Y(t)$ remains orthogonal for all $t \geq t_0$.

Exercise 4. (Euler equations of a rigid body) Consider the system

$$\begin{cases} \dot{y}_1 = \alpha_1 y_2 y_3, & \alpha_1 = 1/I_3 - 1/I_2, \\ \dot{y}_2 = \alpha_2 y_3 y_1, & \alpha_2 = 1/I_1 - 1/I_3, \\ \dot{y}_3 = \alpha_3 y_1 y_2, & \alpha_3 = 1/I_2 - 1/I_1, \end{cases} \tag{2}$$

where $I_1, I_2, I_3 > 0$ are constants.

i) Write the problem in the form $\dot{y} = A(y)y$ and find a first integral $I(y)$ of the system. *Hint:* use Exercise 3.

ii) Which Runge–Kutta methods conserve the first integral that you found in point *i)*? Give the Butcher tableaux of two examples.

iii) Show that another first integral is given by

$$H(y) = \frac{1}{2}\left(\frac{y_1^2}{I_1} + \frac{y_2^2}{I_2} + \frac{y_3^2}{I_3}\right).$$

Exercise 5. (Numerical integration of Euler equations of a rigid body) Consider system (2) with initial condition $y(0) = \begin{pmatrix}\cos(1.1) & 0 & \sin(1.1)\end{pmatrix}^\top$ and constants given by $I_1 = 2, I_2 = 1, I_3 = 2/3$. In the previous exercise we saw that $I_1(y) = y_1^2 + y_2^2 + y_3^2$ and $I_2(y) = \frac{1}{2}\left(\frac{y_1^2}{I_1} + \frac{y_2^2}{I_2} + \frac{y_3^2}{I_3}\right)$ are first integrals of the system.

i) Set the step size $h = 0.5$ and the final time $T = 200$ and solve the problem using the implicit midpoint rule. Verify that the numerical solution lies at the intersection of the sphere $I_1(y) = I_1(y_0)$ and the ellipsoid $I_2(y) = I_2(y_0)$. In order to display the sphere and the ellipsoid use the function `isosurfaces.m` which is on the Moodle page.

ii) Repeat point *i)* using the explicit midpoint rule and verify that the numerical solution leaves the manifolds defined by $I_1(y) = I_1(y_0)$ and $I_2(y) = I_2(y_0)$.

iii) Plot the errors $|I_1(y_n) - I_1(y_0)|$ and $|I_2(y_n) - I_2(y_0)|$ given by both methods at every step n.
Remark: for the implicit midpoint rule the error should be of the order of the machine precision, i.e., $\sim 10^{-16}$. However, in order to see that, the tolerance used to solve the implicit system must be of the order of the machine precision too.