Series 4

Exercise 1. (Resolvent)

Consider the system of differential equations in two dimensions

$$y'(t) = A(t)y(t), \qquad y(t_0) = y_0, \qquad \text{with} \qquad A(t) = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}.$$

i) Compute the resolvent $R(t;t_0)$.
Recall: the resolvent $R(t;t_0)$ has the form $R(t;t_0) = \begin{pmatrix} R_1(t;t_0) & R_2(t;t_0) \end{pmatrix}$ where the columns are defined as the solutions of the systems $R_i'(t;t_0) = A(t)R_i(t;t_0),\ R_i(t_0;t_0) = e_i$ for $i = 1, 2$, with (e_1, e_2) the canonical basis of $\mathbb{R}^2$.

ii) Compute the matrix

$$e^{\int_{t_0}^t A(s)ds},$$

and show that

$$y(t) = R(t;t_0)y_0 \neq e^{\int_{t_0}^t A(s)ds} y_0.$$

Recall: the exponential of a matrix B is defined as $e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}$.

Hence, $e^{\int_{t_0}^t A(s)ds} y_0$ is not a solution in general. What do we further need to assume in order for $e^{\int_{t_0}^t A(s)ds} y_0$ to be a solution?

Exercise 2. Recall the definition of a collocation method. Show that if $f(t, y)$ satisfies a Lipschitz condition in y, a polynomial of degree s satisfying the $s+1$ nonlinear conditions

$$u(t_0) = y_0, \qquad u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)), \qquad \text{for all } i = 1, \ldots, s,$$

exists for h small enough.
Hint: use an argument similar to Exercise 4 of Series 2.

Exercise 3. Consider a Runge–Kutta method with coefficients (a_{ij}, b_i, c_i), $i, j = 1, \ldots, s$. Let $0 < c_2 < c_3 < \cdots < c_{s-1} < 1$ and b_1, b_s be given. Further, assume that $c_1 = 0, c_s = 1$ and $a_{i1} = b_1, a_{is} = 0$, for $i = 1, \ldots, s$. Show that the remaining coefficients are uniquely determined by $C(s-2), B(s-2)$.
Hint: write linear systems with the remaining coefficients as unknowns.

Exercise 4. Show that the discontinuous collocation method defined in the lecture is equivalent to an s-stage Runge–Kutta method with coefficients given by $c_1 = 0, c_s = 1$, $a_{i1} = b_1, a_{is} = 0$, for $i = 1, \ldots, s$, and

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)d\tau - b_1\ell_j(0), \qquad i = 1, \ldots, s,\ j = 2, \ldots, s-1,$$
$$b_i = \int_0^1 \ell_i(\tau)d\tau - b_1\ell_i(0) - b_s\ell_i(1), \qquad i = 2, \ldots, s-1,$$

where $\ell_j(\tau) = \prod_{k=2, k\neq j}^{s-1} \frac{\tau - c_k}{c_j - c_k}$, defined for $j = 2, \ldots, s-1$, are the Lagrange interpolation polynomials of degree $s-3$.

Recall: $C(s-2)$ holds if and only if the quadrature formula defined by the collocation points c_j and the weights a_{ij}, $j = 1, \ldots, s$, integrates exactly polynomials of degree $s-3$ from 0 to c_i and $B(s-2)$ holds if and only if the quadrature formula defined by the collocation points c_j and the weights b_j, $j = 1, \ldots, s$, integrates exactly polynomials of degree $s-3$ from 0 to 1.
Hint: use Exercise 3.

Exercise 5. Show that the Lobatto IIIB quadrature formula has order $2s-2$ for $s \geq 2$, i.e., that

$$\int_0^1 q(x)\mathrm{d}x = \sum_{i=1}^{s} b_i q(c_i) \qquad \forall\, q \in \mathbb{P}^{2s-3}. \tag{1}$$

Recall: the coefficients c_i, $i = 1, \ldots, s$, and b_1, b_s are the same as for the Lobatto IIIA method defined in Exercise $4(iii)$ of Series 3, while b_i, $i = 2, \ldots, s-1$, are defined as in Exercise 3 by $B(s-2)$.
Hint: you may define and use the polynomial $\tilde{p}_s(x) = \frac{p_s(x)}{x(1-x)}$ where p_s is defined in Exercise $4(iii)$ of Series 3. Moreover, remember that the Lobatto IIIA quadrature formula has order $2s-2$ as well.