Series 3

In the lecture we saw the following result:

Theorem. A collocation method is equivalent to the s-stage Runge–Kutta method with coefficients given by

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\, d\tau, \qquad b_i = \int_0^1 \ell_i(\tau)\, d\tau, \tag{1}$$

where ℓ_i are the Lagrange interpolant polynomials given by

$$\ell_i(\tau) = \prod_{k \neq i} \frac{\tau - c_k}{c_i - c_k}.$$

Note that $\ell_i(c_j) = \delta_{ij}$.

Exercise 1. (Quadrature formulas and collocation methods) Let $0 \leq c_1 < \ldots < c_s \leq 1$ be given. Consider for $a_{ij}, b_j \in \mathbb{R}$, for $i = 1, \ldots, s$, the relations

$$C(q): \qquad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \qquad i = 1, \ldots, s \quad \text{and} \quad k = 1, \ldots, q,$$

$$B(q): \qquad \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \qquad k = 1, \ldots, q.$$

i) Prove that $\sum_{j=1}^{s} \ell_j(\tau) c_j^{k-1} = \tau^{k-1}$ if $k \leq s$.

ii) Show that a collocation method satisfies $C(q)$ and $B(q)$ for $q = s$, i.e., that (1) implies $C(q)$ and $B(q)$ for $q = s$.
Hint: use point *i)*.

iii) Show that $C(q)$ for $q = s$ uniquely determines a_{ij} for $i, j = 1, \ldots, s$ and, similarly, that $B(q)$ for $q = s$ uniquely determines b_i for $i = 1, \ldots, s$, i.e., that $C(q)$ and $B(q)$ for $q = s$ imply (1).
Hint: you may write $\ell_i(\tau) = \sum_{k=1}^{s} \alpha_k \tau^{k-1}$ for some $\alpha_k \in \mathbb{R}$.

Points *ii)* and *iii)* show that (1) and $C(q), B(q)$ for $q = s$ are equivalent.

iv) Show that if $C(q)$ holds for $q = s$ then $\int_0^{c_i} p(\tau) d\tau = \sum_{j=1}^{s} a_{ij} p(c_j)$ for all polynomials p of degree $\deg(p) \leq s - 1$. Similarly, show that if $B(q)$ holds for $q = s$ then $\int_0^1 p(\tau) d\tau = \sum_{i=1}^{s} b_i p(c_i)$ for all polynomials p of degree $\deg(p) \leq s - 1$.
Hint: you may write $p(\tau) = \sum_{k=1}^{s} p_k \tau^{k-1}$ for some $p_k \in \mathbb{R}$.

v) In particular, show that a collocation method is consistent ($\sum_{i=1}^{s} b_i = 1$) and invariant under transformation into autonomous form ($\sum_{j=1}^{s} a_{ij} = c_i$).

Exercise 2. Compute the Runge–Kutta coefficients ($a_{i,j}$ and b_i for $i, j = 1, \ldots, s$) of the collocation methods with $s = 2$ nodes as a function of the nodes c_1 and c_2. Then, using the order conditions, find the nodes c_1 and c_2 such that the method has order 4.

Exercise 3. Let $0 \le c_1 < \ldots < c_s \le 1$ and consider the quadrature formula

$$\int_0^1 f(x)dx \approx \sum_{i=1}^s b_i f(c_i). \tag{2}$$

Show that there exist unique scalars $b_1, \ldots, b_s$ such that the above quadrature formula has order (at least) s, i.e.,

$$\int_0^1 p(x)\mathrm{d}x = \sum_{i=1}^s b_i p(c_i) \quad \forall\, p \in \mathbb{P}^{s-1},$$

where $\mathbb{P}^{s-1}$ is the set of polynomials of degree $s-1$.
Hint: use the Lagrange interpolant polynomials ℓ_i.

Exercise 4. (Gauss, Radau and Lobatto quadrature) Let $0 \le c_1 < \ldots < c_s \le 1$ and consider the quadrature formula (2).

i) Consider the Gauss nodes $c_1, \ldots, c_s$ which are the zeros of

$$p_s(x) = \frac{d^s}{dx^s}(x^s(1-x)^s). \tag{3}$$

(a) Find n and α, β such that $p_s(x) = \widetilde{e}_n^{(\alpha,\beta)}\widetilde{w}^{(\alpha,\beta)}(x)\widetilde{p}_n^{(\alpha,\beta)}(x)$.

(b) Show that

$$\int_0^1 p_s(x)q(x)\mathrm{d}x = 0 \quad \forall\, q \in \mathbb{P}^{s-1}. \tag{4}$$

Hint: use the properties of the Jacobi orthogonal polynomials below.

(c) Prove that 0 and 1 are not zeros of p_s, i.e., that $c_1 \neq 0$ and $c_s \neq 1$.
Hint: use the general Leibniz rule for n-th order derivatives of product of functions.

(d) Prove that the zeros of p_s (c_i, $i = 1, \ldots, s$) are distinct and lie in the open interval $(0, 1)$.
Hint: you may use a contradiction argument and point (b) (assume that the statement is false and find a polynomial q such that (4) is false).

(e) Show that the associated Gauss quadrature formula has order $2s$, i.e., that

$$\int_0^1 q(x)\mathrm{d}x = \sum_{i=1}^s b_i q(c_i) \quad \forall\, q \in \mathbb{P}^{2s-1}. \tag{5}$$

Hint: perform a division of polynomials and write $q(x) = t(x)p_s(x) + r(x)$ for some polynomials t and r.

(f) Show that the $\{b_i\}_{i=1}^s$ corresponding to the Gauss quadrature formula are strictly positive, i.e., that $b_i > 0$ for all $i = 1, \ldots, s$.
Hint: prove that $b_i = \int_0^1 \ell_i^2(x)\mathrm{d}x$.

ii) Consider the Radau nodes $c_1, \ldots, c_s$ which are the zeros of

$$p_s(x) = \frac{d^{s-1}}{dx^{s-1}}(x^{s-1}(1-x)^s). \tag{6}$$

Proceed similarly to point *i)*.

(a) Find n and α, β such that $p_s(x) = \widetilde{e}_n^{(\alpha,\beta)}\widetilde{w}^{(\alpha,\beta)}(x)\widetilde{p}_n^{(\alpha,\beta)}(x)$.

(b) Show that

$$\int_0^1 p_s(x)q(x)\mathrm{d}x = 0 \qquad \forall\, q \in \mathbb{P}^{s-2}. \tag{7}$$

(c) Prove that 0 is not and 1 is a zero of p_s, i.e., that $c_1 \neq 0$ and $c_s = 1$.

(d) Prove that the zeros of p_s (c_i, $i = 1, \dots, s$) are distinct and lie in the interval $(0, 1]$.

(e) Show that the associated Radau quadrature formula has order $2s-1$, i.e., that

$$\int_0^1 q(x)\mathrm{d}x = \sum_{i=1}^{s} b_i q(c_i) \qquad \forall\, q \in \mathbb{P}^{2s-2}. \tag{8}$$

(f) Show that the $\{b_i\}_{i=1}^s$ corresponding to the Radau quadrature formula are strictly positive, i.e., that $b_i > 0$ for all $i = 1, \dots, s$.

iii) Consider the Lobatto IIIA nodes $c_1, \dots, c_s$, which are the zeros of

$$p_s(x) = \frac{d^{s-2}}{dx^{s-2}}(x^{s-1}(1-x)^{s-1}). \tag{9}$$

Proceed similarly to point *i*).

(a) Find n and α, β such that $p_s(x) = \widetilde{e}_n^{(\alpha,\beta)} \widetilde{w}^{(\alpha,\beta)}(x) \widetilde{p}_n^{(\alpha,\beta)}(x)$.

(b) Show that

$$\int_0^1 p_s(x)q(x)\mathrm{d}x = 0 \qquad \forall\, q \in \mathbb{P}^{s-3}. \tag{10}$$

(c) Prove that 0 and 1 are zeros of p_s, i.e., that $c_1 = 0$ and $c_s = 1$.

(d) Prove that the zeros of p_s (c_i, $i = 1, \dots, s$) are distinct and lie in the closed interval $[0, 1]$.

(e) Show that the associated Lobatto IIIA quadrature formula has order $2s-2$, i.e., that

$$\int_0^1 q(x)\mathrm{d}x = \sum_{i=1}^{s} b_i q(c_i) \qquad \forall\, q \in \mathbb{P}^{2s-3}. \tag{11}$$

(f) Show that the $\{b_i\}_{i=1}^s$ corresponding to the Lobatto IIIA quadrature formula are strictly positive, i.e., that $b_i > 0$ for all $i = 1, \dots, s$.
Remark: pay attention that the same argument as for Gauss and Radau nodes cannot be employed, but it has to be adapted.

Jacobi orthogonal polynomials. The polynomials (3), (6) and (9) are related to the Jacobi orthogonal polynomials on $[-1, 1]$ which are defined for $n \in \mathbb{N}$ and $\alpha, \beta > -1$ by

$$p_n^{(\alpha,\beta)}(x) = \frac{1}{e_n^{(\alpha,\beta)} w^{(\alpha,\beta)}(x)} \frac{d^n}{dx^n}\big(w^{(\alpha,\beta)}(x)(1+x)^n(1-x)^n\big),$$

where

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta,$$

and the coefficients $e_n^{(\alpha,\beta)}$ are some normalization scalars. The family of polynomials $\{p_n^{(\alpha,\beta)} \,|\, n \geq 0\}$ is orthogonal with respect to the scalar product with weight $w^{(\alpha,\beta)}(x)$

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)w^{(\alpha,\beta)}(x)dx.$$

Then, one can consider the shifted Jacobi orthogonal polynomials $\widetilde{p}_n^{(\alpha,\beta)}(x) = p_n^{(\alpha,\beta)}(2x-1)$ which are orthogonal on the interval $[0, 1]$ with respect to the shifted weight $\widetilde{w}(x) = w(2x-1)$.