Series 2

Exercise 1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function.

i) By applying the mean value theorem to $\int_{t_0}^{t_0+h} f(s, y(s))\mathrm{d}s$, motivate the θ method

$$y_1 = y_0 + hf\Big(t_0 + \theta h, y_0 + \theta(y_1 - y_0)\Big), \tag{1}$$

for the approximation of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Which methods do we obtain for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$?

ii) Further, another variant of the θ method is given by

$$y_1 = y_0 + h(1-\theta)f(t_0, y_0) + h\theta f(t_0 + h, y_1). \tag{2}$$

Which methods do we obtain for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$?

iii) Show that both versions (1) and (2) are Runge–Kutta methods (give the coefficients in a Butcher tableau).

Exercise 2. Give the order of the methods defined by (1) and (2) in Exercise 1 for the values $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$. Further, show that the Runge's method, given by the Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array},$$

has order 2.
Hint: compare the Taylor series of the numerical and exact solutions or use the order conditions.

Exercise 3. Show that the order p of an s-stage *explicit* Runge–Kutta method satisfies $p \leq s$.
Hint: find a particular problem for which we must have $p \leq s$.

Exercise 4. Show that, if the field $f\colon \mathbb{R}^{1+d} \to \mathbb{R}^d$, $f(t, y)$, satisfies a Lipschitz condition in y, an implicit Runge–Kutta method applied to $y'(t) = f(t, y(t))$ is well defined (for a step size which is small enough), i.e., the nonlinear system defining the method has a unique solution.
Hint: use a fixed-point theorem.

Exercise 5. Consider a Runge–Kutta method with coefficients a_{ij}, b_i, for $i, j = 1, \dots, s$. Suppose that the method has order p for all autonomous systems

$$y'(t) = f(y(t)), \qquad y(0) = y_0.$$

Show that, if the coefficients c_i are defined by $c_i = \sum_{j=1}^{s} a_{ij}$, for $i = 1, \dots, s$, this method has also order p when applied to a non autonomous system

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

Recall: for a consistent method, i.e., with order $p \geq 1$, we have $\sum_{i=1}^{s} b_i = 1$.
Hint: use the transformation $Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix}$.