Series 1

The Kepler problem: discretization and conservation of first integrals

The motion of two bodies which attract each other by gravity is described by the Kepler problem

$$\ddot{q}_1 = -\frac{q_1}{(q_1^2 + q_2^2)^{3/2}}, \qquad \ddot{q}_2 = -\frac{q_2}{(q_1^2 + q_2^2)^{3/2}}, \tag{1}$$

where $(q_1(t), q_2(t))$ denotes the position at time t of the second body relatively to the first body.

This system is equivalent to a Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}(p, q), \qquad \dot{p} = -\frac{\partial H}{\partial q}(p, q), \tag{2}$$

where $p = \dot{q}$ is the velocity and the energy

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}},$$

is a conserved quantity along time, i.e., $H\Big(p(t), q(t)\Big) = const.$

We choose the initial conditions

$$q_1(0) = 1 - e, \quad q_2(0) = 0, \quad \dot{q}_1(0) = 0, \quad \dot{q}_2(0) = \sqrt{\frac{1+e}{1-e}},$$

where $0 \le e < 1$ is a fixed parameter, for instance $e = 0.6$. Then the trajectory is elliptic with eccentricity e and the motion is periodic with period 2π.

Exercise 1. Numerical discretization
Implement the following numerical methods for the Kepler problem:

1. Euler explicit method:

$$q_{n+1} = q_n + h\frac{\partial H}{\partial p}(p_n, q_n), \qquad p_{n+1} = p_n - h\frac{\partial H}{\partial q}(p_n, q_n),$$

2. Euler symplectic method:

$$q_{n+1} = q_n + h\frac{\partial H}{\partial p}(p_n, q_{n+1}), \qquad p_{n+1} = p_n - h\frac{\partial H}{\partial q}(p_n, q_{n+1}),$$

3. Euler implicit method:

$$q_{n+1} = q_n + h\frac{\partial H}{\partial p}(p_{n+1}, q_{n+1}), \qquad p_{n+1} = p_n - h\frac{\partial H}{\partial q}(p_{n+1}, q_{n+1}),$$

4. Implicit midpoint rule:

$$\begin{aligned} q_{n+1} &= q_n + h\frac{\partial H}{\partial p}\left(\frac{1}{2}(p_n + p_{n+1}), \frac{1}{2}(q_n + q_{n+1})\right), \\ p_{n+1} &= p_n - h\frac{\partial H}{\partial q}\left(\frac{1}{2}(p_n + p_{n+1}), \frac{1}{2}(q_n + q_{n+1})\right). \end{aligned}$$

In each case, integrate over 20 periods with step size $h = 2 \cdot 10^{-4}$ and plot the obtained trajectory $\{q_n\,;\, n = 0, 1, 2, \ldots\}$.
Hint: in order to implement implicit methods you can employ fixed-point iterations or Newton method with tolerance tol $= 10^{-4}$ or use the functions `fsolve` in MATLAB or `scipy.optimize.fsolve` in PYTHON.

Exercise 2. Conservation of energy
In each case, check whether the energy $H(p_n, q_n)$ is conserved, by plotting $H(p_n, q_n)$ as a function of time $t = nh$.

By trying different step sizes h, find numerically for each method what is the corresponding behavior where $t = nh$:

$$H(p_n, q_n) - H(p_0, q_0) = \mathcal{O}(ht), \quad \mathcal{O}(h^2 t), \quad \mathcal{O}(h), \quad \mathcal{O}(h^2).$$

Exercise 3. Angular momentum conservation
The exact solution of the Kepler problem also conserves the angular momentum defined by

$$L(p, q) = q_1 p_2 - q_2 p_1.$$

Verify that L is a first integral of the system. Similarly to Exercise 2, check whether L is conserved by the considered numerical discretizations.

Exercise 4. Global error

We wish to study the behavior of the global error, i.e.,

$$\|q_n - q(nh)\| \text{ and } \|p_n - p(nh)\| \text{ for } n = 1, 2, 3, \ldots$$

The motion is periodic with period 2π. Then for a stepsize of the form $h = 2\pi/r$, where r is a positive integer, we have $q_{kr} \approx q(k2\pi) = q_0$ and $p_{kr} \approx p(k2\pi) = p_0$ for $k = 1, 2, 3, \ldots$ Find numerically the behavior of $\|q_{kr} - q_0\|$ and $\|p_{kr} - p_0\|$, $k = 1, 2, 3, \ldots$ for each method, among:

$$\mathcal{O}(ht^2), \quad \mathcal{O}(h^2t^2), \quad \mathcal{O}(ht) \quad \text{or} \quad \mathcal{O}(th^2),$$

where $t = k2\pi$ is the time.
Hint: to study the convergence w.r.t. h set $t = 2\pi$ and $h = 10^{-5}\pi \cdot 2^m$ for $m = 0, \ldots, 15$ and to study the convergence w.r.t. t set $h = 10^{-4}\pi$ and $t = 2\pi \cdot 2^m$ with $m = 1, \ldots, 8$.

Collect your results of Exercise 2, 3, 4 in the table below.

method	error in H	error in L	global error
explicit Euler			
symplectic Euler			
implicit Euler			
midpoint rule			

Exercise 5. Linear energy drift
Consider a Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q), \qquad \dot{q} = \frac{\partial H}{\partial p}(p, q),$$

where H is Lipschitz. Rewrite the system in the form $\dot{y} = J^{-1}\nabla H(y)$ where $y = (p, q)^T$ (find the matrix J).
Show that for a numerical method of order p integrated over $n = 1, \ldots N$ steps of length h, one has for $t = nh$,

$$H(y_n) - H(y_0) = \mathcal{O}(th^p).$$