Series 13

Exercise 1. Show that a Runge–Kutta method has second order of accuracy if and only if its stability function satisfies $R(z) = 1 + z + \frac{1}{2}z^2 + \mathcal{O}(z^3)$.

Exercise 2. **(Chebyshev polynomials)** For $s \in \mathbb{N}$ and $x \in [-1, 1]$ define

$$T_s(x) = \cos(s \arccos(x)).$$

i) Show that the following recurrence relation holds

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_s(x) &= 2xT_{s-1}(x) - T_{s-2}(x), \qquad s \geq 2. \end{aligned}$$

Hint: use the change of variables $\varphi = \arccos(x)$.
Remark: due to the recurrence relation, T_s is a polynomial of degree s with leading term $2^{s-1}x^s$. Moreover, the functions T_s can now be defined on the whole $\mathbb{R}$ and not only on $[-1, 1]$.

ii) Show that the local extrema and the zeros of T_s are given by

$$\begin{aligned} T_s\left(\cos\left(\frac{k\pi}{s}\right)\right) &= (-1)^k, \qquad k = 0, \dots, s, \\ T_s\left(\cos\left(\frac{(2k+1)\pi}{2s}\right)\right) &= 0, \qquad k = 0, \dots, s-1. \end{aligned}$$

iii) Show that the polynomials T_s are orthogonal on $[-1, 1]$ with respect to the weight function $1/\sqrt{1-x^2}$. In particular, prove that

$$\int_{-1}^{1} T_s(x)T_r(x)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \begin{cases} 0 & \text{if } s \neq r \\ \pi & \text{if } s = r = 0 \\ \frac{\pi}{2} & \text{if } s = r \neq 0 \end{cases}.$$

iv) Prove by induction that

$$T_s'(1) = s^2 \qquad \text{and} \qquad T_s''(1) = \frac{1}{3}s^2(s^2 - 1).$$

Exercise 3. **(Runge–Kutta–Chebyshev method)** Consider the system of differential equations $\dot{y} = f(y)$ with $y(0) = y_0$ and the s-stage Runge–Kutta–Chebyshev (RKC) method defined by

$$\begin{aligned} g_0 &= y_0, \\ g_1 &= y_0 + \frac{\Delta t}{s^2} f(g_0), \\ g_i &= \frac{2\Delta t}{s^2} f(g_{i-1}) + 2g_{i-1} - g_{i-2}, \qquad i = 2, \dots, s, \\ y_1 &= g_s. \end{aligned} \tag{1}$$

i) Show that if $f(y) = \lambda y$ then the RKC method after one step gives

$$y_1 = T_s\left(1 + \frac{\Delta t \lambda}{s^2}\right) y_0,$$

where T_s is the Chebyshev polynomial of degree s.

ii) Rewrite the RKC method using the explicit Runge–Kutta notation. In particular, find a recurrence relation for the coefficients a_{ij} for $i = 1, \ldots, s$ and $j = 0, \ldots, i-1$ such that

$$g_i = y_0 + \Delta t \sum_{j=0}^{i-1} a_{ij} f(g_j), \qquad i = 1, \ldots, s,$$

where $k_j = f(g_j)$.

iii) Prove by induction that

$$\sum_{j=0}^{i-1} a_{ij} = \frac{i^2}{s^2}, \qquad i = 1, \ldots, s.$$

iv) Show that the RKC method has order at least $p = 1$.
Hint: notice that using the Runge–Kutta notation $b_j = a_{sj}$ for $j = 0, \ldots, s-1$.

v) Show that the RKC method has not order $p = 2$.
Hint: consider $f(y) = \lambda y$ and prove that $y_1 - y(\Delta t)$ is not $\mathcal{O}((\Delta t)^3)$.

vi) If we apply the RKC method to the nonautonomous system $\dot{y} = h(t, y)$, how should we choose the c_i for $i = 1, \ldots, s$ of the Runge–Kutta formulation?

Exercise 4. (Stability analysis of the Runge–Kutta–Chebyshev method) Consider the RKC method (2) applied to the test equation $\dot{y} = \lambda y$ with $y(0) = y_0$ and $\lambda < 0$.

i) What is the restriction on the step size Δt in order for the method to be stable, as a function of the number of stages s?

ii) If Δt is fixed, how do you choose the number of stages s in order for the method to be stable?
Remark: notice that varying the number of stages s we obtain an unconditionally stable explicit Runge–Kutta method.

iii) Based on the linear stability analysis, compare the cost of the RKC method and the explicit Euler method. Fix the final time T and for both methods set the maximal step size Δt such that the methods are stable. Measure the cost as the number of function evaluations.