Series 12

Exercise 1. Consider a collocation method with $s = 2$ nodes. Which condition do the nodes c_1 and c_2 need to satisfy in order for the method to be A-stable?

Exercise 2. Consider the linear system $\dot{y} = Ay$ with $A \in \mathbb{C}^{d\times d}$ and assume that

$$\operatorname{Re} \langle y, Ay\rangle \leq 0 \qquad \text{for all } y \in \mathbb{C}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product on $\mathbb{C}^d$, i.e., $\langle x, y\rangle = x^\top \bar{y}$ for all $x, y \in \mathbb{C}^d$, where $\bar{y}$ is the complex conjugate of y. Moreover, define the norm for $C \in \mathbb{C}^{d\times d}$ associated with the scalar product

$$\|C\| \coloneqq \sup_{u,v\in\mathbb{C}^d, \|u\|\leq 1, \|v\|\leq 1} |\langle u, Cv\rangle|.$$

Finally, let $R(z)$ be the stability function of an A-stable Runge–Kutta method. Hence, $|R(z)| \leq 1$ for all $z \in \mathbb{C}^-$ and $R(z)$ is holomorphic on $\mathbb{C}^-$.

i) Show that the norm of the solution $\|y(t)\|$ is a decreasing function in time.

Assume first that A is normal, i.e., $AA^* = A^*A$ where A^* is the conjugate transpose of A. In this case, A can be diagonalized by a unitary matrix Q such that $A = QDQ^*$ where D is diagonal and $QQ^* = Q^*Q = I_d$.

ii) Show that the eigenvalues of A belong to $\mathbb{C}^-$.

iii) Show that

$$\|R(A)\| \leq \sup_{\operatorname{Re} z\leq 0} |R(z)|.$$

Hint: use, without proving it, that since $R(z)$ is holomorphic on $\mathbb{C}^-$, then it can be written as a power series, so it holds

$$R(A) = R(QDQ^*) = QR(D)Q^*,$$

where the stability function R is then applied to each component of the diagonal of D.

Assume now that A is a general matrix and define the matrix function $\mathcal{A}\colon \mathbb{C} \to \mathbb{C}^{d\times d}$

$$\mathcal{A}(\omega) \coloneqq \frac{\omega}{2}(A + A^*) + \frac{1}{2}(A - A^*).$$

Moreover, for any fixed vectors $u, v \in \mathbb{C}^d$ define the function $\varphi\colon \mathbb{C} \to \mathbb{C}$

$$\varphi(\omega) \coloneqq \langle u, R(\mathcal{A}(\omega))v\rangle .$$

iv) Show that $\mathcal{A}(\omega)$ satisfies for all $\operatorname{Re}\omega \geq 0$

$$\operatorname{Re} \langle y, \mathcal{A}(\omega)y\rangle \leq 0 \qquad \text{for all } y \in \mathbb{C}^d,$$

v) Show that the eigenvalues of $\mathcal{A}(\omega)$ belong to $\mathbb{C}^-$ for all $\operatorname{Re}\omega \geq 0$.

vi) Show that $\mathcal{A}(ix)$ is normal for all $x \in \mathbb{R}$.

vii) Show that

$$\|R(A)\| \leq \sup_{\operatorname{Re} z \leq 0} |R(z)|.$$

Hint: use, without proving it, that since $R(z)$ is holomorphic on $\mathbb{C}^-$, employing the Jordan canonical form and by point *v*), then φ is a rational holomorphic function on $\mathbb{C}^+$. Hence, applying the Phragmén–Lindelöf theorem, which is an extension of the maximum principle for holomorphic functions, the function φ satisfies

$$|\varphi(1)| \leq \sup_{x \in \mathbb{R}} |\varphi(ix)|.$$

viii) Deduce that the numerical solution is contractive, i.e., $\|y_{n+1}\| \leq \|y_n\|$, and thus the numerical method preserves the property *i*) of the system.
Hint: use that $y_{n+1} = R(hA)y_n$.

Exercise 3. Show that the stability functions of Gauss, Radau and Lobatto IIIA collocation methods with s collocation points are Padé approximations. In particular, prove that

$$R_{\text{Gauss}}(z) = R_{s,s}(z),$$
$$R_{\text{Radau}}(z) = R_{s-1,s}(z),$$
$$R_{\text{Lobatto IIIA}}(z) = R_{s-1,s-1}(z).$$

Remark: Notice that the stability functions of Gauss and Lobatto IIIA collocation methods are therefore given in the diagonal of the following table, where the Padé approximation $R_{kj}(z)$ is computed for $j, k = 0, 1, 2$.

	$k = 0$	$k = 1$	$k = 2$
$j = 0$	1	$1 + z$	$1 + z + \frac{1}{2}z^2$
$j = 1$	$\frac{1}{1 - z}$	$\frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$	$\frac{1 + \frac{2}{3}z + \frac{1}{6}z^2}{1 - \frac{1}{3}z}$
$j = 2$	$\frac{1}{1 - z + \frac{1}{2}z^2}$	$\frac{1 + \frac{1}{3}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2}$	$\frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2}$

Moreover, the stability functions of Radau collocation methods are given in the subdiagonal of the same table.

Exercise 4. Let φ_h denote the exact flow of a system of differential equations $y'(t) = f(y(t))$ with $y(t_0) = y_0$. Consider a numerical method Φ_h of order p and let $y_2 = (\Phi_h \circ \Phi_h)(y_0)$, $\omega = \Phi_{2h}(y_0)$ and $z_2 = (2^p y_2 - \omega)/(2^p - 1)$.

i) Show that

$$y(t_0 + 2h) - \omega = 2^{p+1} C(y_0) h^{p+1} + \mathcal{O}(h^{p+2}),$$

where $C(y_0)$ is a constant dependent on the initial condition.

ii) Show that for the same constant $C(y_0)$

$$\varphi_h(y_1) - y_2 = C(y_0) h^{p+1} + \mathcal{O}(h^{p+2}).$$

iii) Show that for the same constant $C(y_0)$

$$\varphi_h(y_1) = \varphi_{2h}(y_0) - C(y_0) h^{p+1} + \mathcal{O}(h^{p+2}).$$

iv) Combining points *ii)* and *iii)* deduce that for the same constant $C(y_0)$

$$y(t_0 + 2h) - y_2 = 2C(y_0)h^{p+1} + \mathcal{O}(h^{p+2}).$$

v) Combining points *i)* and *iv)* deduce that

$$y(t_0 + 2h) - z_2 = \mathcal{O}(h^{p+2}),$$

which defines the accelerated method named Richardson extrapolation.