Series 10

Exercise 1. (Curtis and Hirschfelder, 1952) Consider the problem

$$\dot{y}(t) = f(t, y(t)) = \lambda(y(t) - \cos(t)), \qquad y(0) = y_0, \tag{1}$$

where $\lambda < 0$.

i) Compute the exact solution $y_{\text{ex}}(t)$ of problem (1).
ii) Perform a linear stability analysis and find the time-step restriction for the explicit Euler method applied to the linearized problem. What happens if $|\lambda|$ becomes large?
iii) Repeat point *ii)* for the implicit Euler method. What is the time-step restriction now?

We verify that the figures seen in class are indeed what we obtain numerically.

iv) Apply the explicit Euler method to problem (1) with $y_0 = 0$, $\lambda = -2000$, final time $T = 0.05$ and using step sizes $h = 1.9/|\lambda|$, $h = 2/|\lambda|$, $h = 2.1/|\lambda|$. Verify that the oscillations decrease, stay constant and increase in amplitude, respectively.
v) Apply the implicit Euler method, the implicit midpoint rule and the trapezoidal rule to problem (1) with $y_0 = 0$, $\lambda = -2000$, final time $T = \pi/2$ and using step size $h = 200/|\lambda|$. What happens to the oscillations?

Let $\alpha \in [0, 1]$ and $f_\alpha(t, y) = \alpha f(t, y) + (1 - \alpha) f(t, y_{\text{ex}}(t))$, then the solution of

$$y'(t) = f_\alpha(t, y(t)), \qquad y(0) = y_0, \tag{2}$$

is still $y_{\text{ex}}(t)$.

vi) Apply the explicit Euler method to problem (2) with $y_0 = 0$, $\lambda = -2000$, final time $T = 0.01$ and using step size $h = 2/|\lambda|$ for different values of $\alpha = 1/3, 2/3, 1$. Verify that the amplitude of the oscillations depends on α and therefore stiffness is a property of problem (2) and not of the exact solution $y_{\text{ex}}(t)$.

Exercise 2. Consider the θ-method for $\theta \in [0, 1]$

$$y_{n+1} = y_n + \theta h f(t_n, y_n) + (1 - \theta) h f(t_{n+1}, y_{n+1}). \tag{3}$$

What is the time-step restriction for the method (4) applied to the test problem $\dot{y} = \lambda y$, $y(0) = y_0$? For which va$^{l}uesof\theta$ is the method A-stable?

Exercise 3. (Eigenvalues of the discrete Laplacian) Consider the matrix $A \in \mathbb{R}^{N \times N}$ used for the discretization of the Laplacian operator with Dirichlet boundary conditions in one dimension

$$A = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & \end{pmatrix}.$$

The aim of this exercise is to compute the eigenvalues and the eigenvectors of A.

i) Let (μ, v) be a couple eigenvalue-eigenvector of the matrix $B = A + 2I_N$. Show that $(\mu - 2, v)$ is a couple eigenvalue-eigenvector of A and that the components $(v_1, \ldots, v_N)$ of v satisfy the relation

$$v_0 = 0, \qquad v_{j-1} + v_{j+1} = \mu v_j, \quad j = 1, \ldots, N, \qquad v_{N+1} = 0.$$

ii) Using the ansatz $v_j = x^j$ deduce the relation

$$v_j = C(x_1^j - x_2^j), \qquad j = 1, \ldots, N,$$

for a constant $C \in \mathbb{C}$ and where $x_1, x_2 \in \mathbb{C}$ satisfy

$$x_1 + x_2 = \mu, \qquad x_1 x_2 = 1 \qquad \text{and} \qquad \left(\frac{x_1}{x_2}\right)^{N+1} = 1.$$

iii) Find the values of x_1, x_2 and μ.

iv) Deduce that the eigenvalues of A are

$$\lambda_k = 2\cos\left(\frac{k\pi}{N+1}\right) - 2, \qquad k = 1, \ldots, N,$$

with associated eigenvectors $v^{(k)} = (v_1^{(k)}, \ldots, v_N^{(k)})^\top$, $k = 1, \ldots, N$, given by

$$v_j^{(k)} = \widetilde{C} \sin\left(\frac{jk\pi}{N+1}\right), \qquad j = 1, \ldots, N,$$

for a constant $\widetilde{C} \in \mathbb{C}$.

Exercise 4. Let $\Omega \subset \mathbb{R}$ be a bounded open set and let $u\colon \Omega \to \mathbb{R}$ be a smooth function. Moreover, let $x_i \in \Omega$ and $\Delta x > 0$ be such that $[x_{i-1}, x_{i+1}] \subset \Omega$ with $x_{i\pm1}$ given by $x_{i\pm1} = x_i \pm \Delta x$.

i) Show that

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{(\Delta x)^2} = u''(x_i) + \mathcal{O}\left((\Delta x)^2\right). \tag{4}$$

ii) What is the minimum regularity of u in order for (7) to hold?

iii) What happens to the approximation (7) if u is only $\mathcal{C}^2(\Omega)$?