Properties of the Chebyshev polynomials: (see exercises)

(i) $|T_s(x)| \leq 1 \quad \forall \; x \in [-1,1]$.

(ii) $T_s\left(\cos\left(\frac{k\pi}{s}\right)\right) = (-1)^k \quad , k = 0, 1, \dots, s$

$T_s\left(\cos\left(\frac{(2k+1)\pi}{2s}\right)\right) = 0 \quad , k = 0, 1, \dots, s-1$

(iii) $\{T_s(x)\}_{s\geq 0}$ are orthogonal on $[-1,1]$

to the weight fct $\frac{1}{\sqrt{1-x^2}}$, i.e.

$$\int_{-1}^{1} \frac{T_j(x)\, T_i(x)}{\sqrt{1-x^2}}\, dx = 0 \quad , \text{if } i \neq j.$$

(iv) $T_s'(1) = s^2$ and $T_s''(1) = \frac{1}{3} s^2 (s^2 - 1)$

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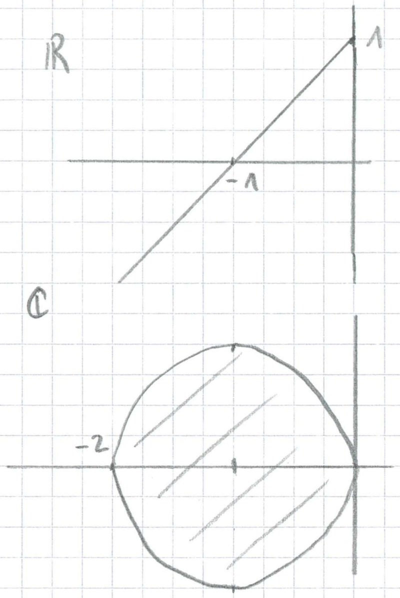
## Examples:

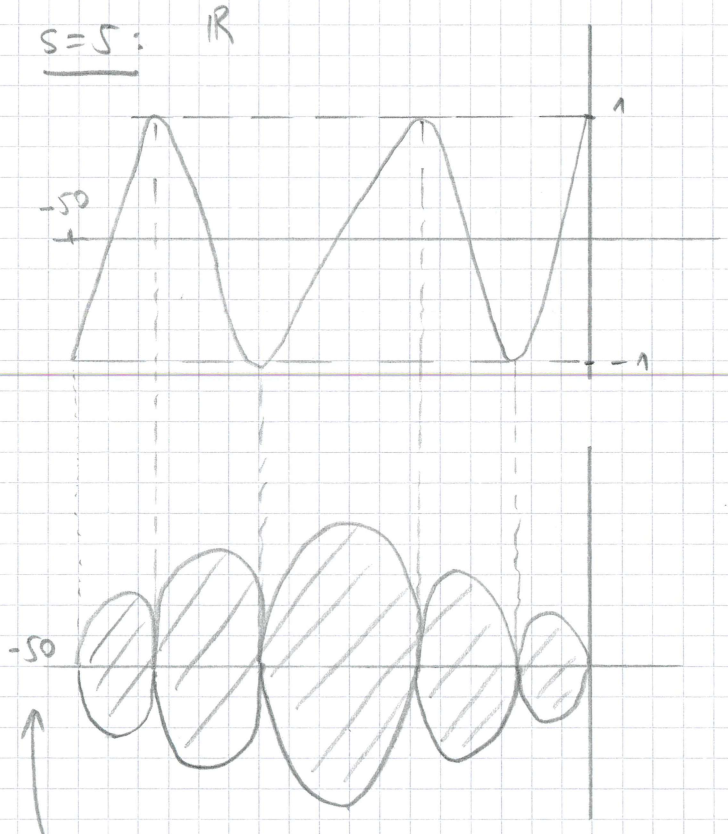
**s=1:**




$R_1(z) = 1+z$

stability domain

**s=5:**




$R_5(z) = 1+z+\alpha_2 z^2+\alpha_3 z^3+\alpha_4 z^4+\alpha_5 z^5$

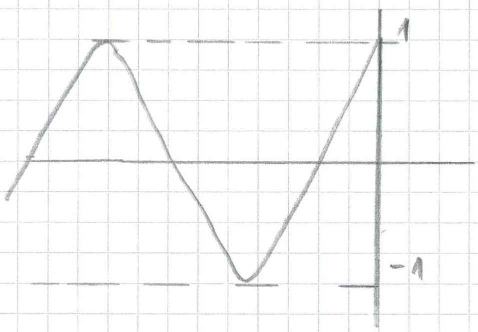
$50 = 2\cdot 25 = 2\cdot 5^2 = 2\cdot s^2.$

<u>Rem:</u> At $z_1, z_2, \dots, z_s$ we have no "width" in the stability region along $\mathrm{Im}(z)$.

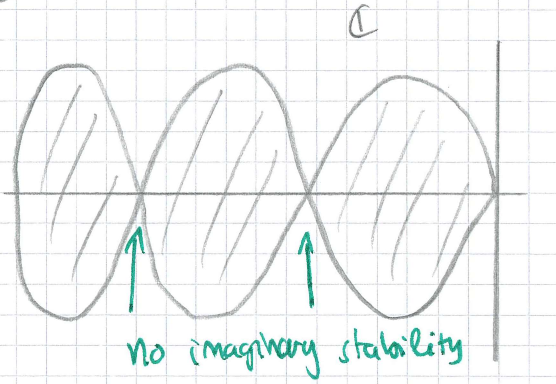
This may be a problem

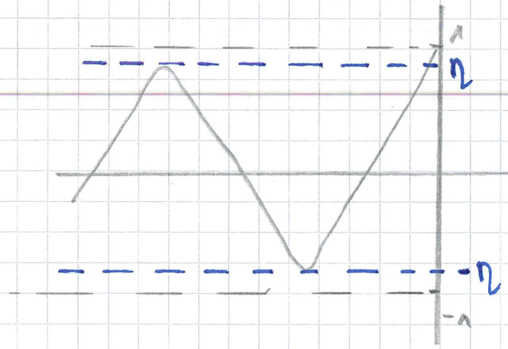
$\Rightarrow$ solution given by damped Chebyshev polynomials (<u>damping</u>).

<u>Damping</u> : Ask for $|R_s(z)| < \eta < 1$ for $z \in [-\tilde{\ell}_s, -\delta_{s,\eta}]$.

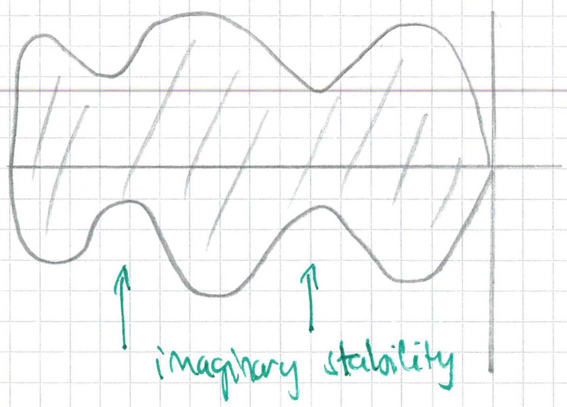



no damping




damping




To achieve that, define

$$\hat{R}_s(z) = \frac{T_s(w_0 + w_1 z)}{T_s(w_0)} \quad \text{with} \quad w_0 = 1 + \frac{\eta}{s^2}, \quad w_1 = \frac{T_s(w_0)}{T_s'(w_0)}.$$

# Runge-Kutta-Chebyshev (RKC)

We need to build a numerical integrator for ODEs with $R_s$ as its stability function.

<u>Defn:</u> (Runge-Kutta-Chebyshev method)

$$g_0 = y_0$$
$$g_1 = y_0 + \frac{h}{s^2} f(g_0)$$
$$g_j = 2\frac{h}{s^2} f(g_{j-1}) + 2g_{j-1} - g_{j-2}, \quad j = 2,3,\dots,s$$
$$y_1 = g_s$$

<u>Theorem:</u> The Runge-Kutta-Chebyshev method has $R_s(z) = T_s\left(1 + \frac{z}{s^2}\right)$ as stability function.

<u>Proof:</u> see exercises.

---

<u>Rem:</u> 1) $R_s(z) = 1 + z + \mathcal{O}(z^2) \Rightarrow$ RKC has order 1.

2) With $s$ fct evaluations we get a factor of $s^2$ in the stability "width".

3) Consider $y' = f(y)$

and $\rho = \max_i |\lambda_i(f'(y))|$ with $\lambda_1, \dots, \lambda_d$ real and negative
(stiffness index)

Then $|R_s(z)| \leq 1 \quad \forall z \in [-2s^2, 0]$

For $h\rho \leq 2s^2 \iff h \leq \underbrace{\frac{2}{\rho}}_{\text{restriction for Explicit Euler}} s^2$ (time step restriction)

the RKC is stable.

4) Actually, we don't have time step restrictions:
For any given $h$ just choose $s$ s.t.

$$s \geq \sqrt{\frac{hs}{2}} \quad \text{for stability.}$$

5) For $y' = f(t,y)$ non-autonomous:

$$c_0 = 0, \quad c_1 = \frac{1}{s^2}$$

$$c_j = \frac{2}{s^2} + 2c_{j-1} - c_{j-2}, \quad j = 2, \dots, s$$

## Efficiency of the RKC method

For one step of RKC: $s$ fct evaluations
How does this compare to $s$ steps of explicit Euler?

Suppose that Tol = desired accuracy.

1) $\text{Tol} > \frac{2}{S}$ (stiff)

EE: We cannot set $h_{EE} = \text{Tol} > \frac{2}{S}$

$$\Rightarrow \text{set } h_{EE} = \frac{2}{S}.$$

RKC: Choose $h_{RKC} = \text{Tol}$ and $s = \left\lceil \sqrt{\frac{h_{RKC} \cdot S}{2}} \right\rceil = \left\lceil \sqrt{\frac{\text{Tol} \cdot S}{2}} \right\rceil$

for stability

To reach time $t = \text{Tol}$, we have

$$\text{Cost}_{RKC} = s = \left\lceil \sqrt{\frac{\text{Tol} \cdot S}{2}} \right\rceil.$$

For EE, to reach $t = Tol$, we need $\left\lceil \frac{Tol}{h_{EE}} \right\rceil$ steps:

$$Cost_{EE} = \left\lceil \frac{Tol}{h_{EE}} \right\rceil = \left\lceil \frac{Tol \cdot \rho}{2} \right\rceil = \mathcal{O}(Cost_{RKC}^2)$$

$$\Rightarrow \text{Speed-up} = \mathcal{O}\left(\sqrt{\frac{Tol \cdot \rho}{2}}\right) \gg 1.$$

2) $Tol \leq \frac{2}{\rho}$

In this case $h_{EE} = Tol$ is ok, but RKC can run on $s = 1$ stages $\Rightarrow$ RKC $\equiv$ EE.

Conclusion: RKC method is good for diffusive problems.

Algorithm: $h/s$ adaptive RKC method:

(i) Estimate the spectral radius

$$\rho = \max_i \left| \lambda_i \left( \frac{\partial f}{\partial y}(t_n, y_n) \right) \right|$$

(ii) Choose $s = \left\lceil \sqrt{\frac{h_n \cdot \rho}{2}} \right\rceil$ for stability.

(iii) Compute $y_{n+1}$ with RKC on $s$ stages.

(iv) Estimate $E_{n+1}$ (non-trivial), determine $h_{n+1}$ to match $Tol$.

(v) Usual accept / reject machinery.

Rem: RKC is the simplest method of the family of explicit stabilized Runge-Kutta methods:

ROCK 2/4 (Abdulle, Medovikov)

RKC 1/2 (Houwen, Sommeijer, Verwer)

Recent developments:

(Abdulle & Rosilho de Souza 2020)

(Abdulle & Blumenthal 2013)

Rem: Higher order methods much more difficult to construct (used & efficient in practice).

$$\underbrace{1 + z + \frac{z^2}{2!}}_{\text{2nd order}} + \underbrace{\sum_{i=3}^{s} \alpha_i z^i}_{\text{~~want to optimize but unfortunately~~ no Chebyshev polynomials}}$$

Rem: Stabilized multilevel Monte Carlo method for stiff stochastic differential equations (see Abdulle & Blumenthal 2013).

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## Summary

| | explicit | implicit | Conserv. polyn. inv. | symmetric | symplectic | good for Geom. Int. | good for stiff prob. |
|---|---|---|---|---|---|---|---|
| explicit methods RKC | yes | no | lin. | no | no | no | yes (Cheb.) no (general) |
| Radau $y_1 = y_0 + hf(y_1)$ | no | yes | lin. | no | no | no | yes |
| Gauss $y_1 = y_0 + h f\left(\frac{y_0+y_1}{2}\right)$ | no | yes | quad. | yes | yes | yes | no (not L-stable) |
| Lobatto IIIA $y_1 = y_0 + \frac{h}{2} f(y_0) + \frac{h}{2} f(y_1)$ | no | yes | lin. | yes | no | no / yes (symm. really good) | no (not L-stable. |
| Partitioned methods: Lobatto IIIA - IIIB, Symplectic Euler | yes / no $H = T + U$ separable Hamiltonian systems | yes / no | quad. | no | yes | yes | no |