AB

Lemma: A RK method is A-stable if and only if its stability function satisfies:

(i) $|R(iy)| \leq 1 \quad \forall y \in \mathbb{R}$

(ii) $R(z)$ analytic in $\mathbb{C}^-$.

Proof: Assume A-stability.

$\Rightarrow$: (i) follows by definition

(ii) follows since $|R(z)| \leq 1$ on $\mathbb{C}^-$ and R is rational

$\Rightarrow$ R has no poles on $\mathbb{C}^-$

$\Rightarrow$ R is analytic on $\mathbb{C}^-$.

$\Leftarrow$: Assume (i) and (ii)

Recall: If f analytic and $\Omega \subset \mathbb{C}$ open set then either max of $|f|$ on the boundary of Ω or f constant.

maximum principle

Now, setting $\Omega = \mathbb{C}^-$, R analytic by assumption and not constant

$\Rightarrow$ max on imaginary axis (boundary of $\mathbb{C}^-$)

Let $y^* \in \mathbb{R}$ s.t. $|R(iy^*)|$ max of $|R|$.
By assumption $|R(iy)| \leq 1 \ \forall y \in \mathbb{R}$, and thus,

$$|R(z)| \leq |R(iy^*)| \leq 1 \quad \forall z \in \mathbb{C}^-$$

$\Rightarrow$ method A-stable. #

Defn.: (L-stable)

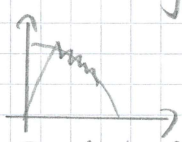
A RK method is L-stable if it is A-stable and if additionally

$$\lim_{z \to -\infty} R(z) = 0,$$

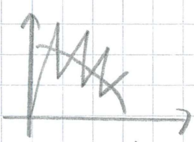
where R is its stability function.

Rem.: ① L-stability is associated with "damping" of oscillations.

Curtis-Hirschfelder problem



Implicit Euler
L-stable



Implicit Midpoint Rule
not L-stable

Gauss methods are not appropriate for stiff problems in general, Radau methods are.

② If $S = \mathbb{C}^- \Rightarrow$ method is not L-stable. (but A-stable)

Indeed, in this case $|R(iy)| = 1$ for $y \in \mathbb{R}$ and for rational fct: $\lim_{z \to -\infty} R(z) = \lim_{y \to \infty} R(iy) = 1$. $\Rightarrow$ not L-stable.

4. Stability functions of collocation methods

Consider $0 \le c_1 < c_2 < \dots < c_s \le 1$ collocation points
and consider

$$T(t) = \frac{1}{s!} \prod_{i=1}^{s} (t - c_i)$$

$\Rightarrow$ 1) $T(c_i) = 0$, $i = 1, 2, \dots, s$
2) $\frac{d^s}{dt^s} T(t) = 1$.

Theorem: The stability function of a collocation method based on $c_1, c_2, \dots, c_s$ is given by

$$R(z) = \frac{T^{(s)}(1) + z\,T^{(s-1)}(1) + \dots + z^{s-1} T'(1) + z^s T(1)}{T^{(s)}(0) + z\,T^{(s-1)}(0) + \dots + z^{s-1} T'(0) + z^s T(0)}.$$

Proof: Apply the collocation method to $y' = \lambda y$, and WLOG set $t_0 = 0$, $h = 1$ $(z = \lambda)$

$$\Rightarrow \begin{cases} u(0) = y_0 \\ u'(c_i) = z\,u(c_i) & (= f(u(c_i))) \\ y_1 = u(1) \end{cases}$$

It follows that $R(z) = \frac{u(1)}{u(0)}$. $\left(= \frac{y_1}{y_0} \text{, since } R(z) y_0 = y_1 \right)$

Consider the mismatch function $u'(t) - z\,u(t)$, which is zero on c_i: $u'(c_i) - z\,u(c_i) = 0$ (see above)

indep. of t

$$\Rightarrow u'(t) - z\,u(t) = K\,T(t) \qquad (*)$$

both polynomials of degree s with the same zeros

Differentiate $(*)$ $u' = zu + KT$

$$\Rightarrow u'' = zu' + KT'$$
$$= z^2 u + zKT + KT'$$

$$\Rightarrow \quad u''' = z^2 u' + zKT' + KT''$$
$$= z^3 u + K(z^2 T + zT' + T'')$$

$\vdots$

$$\Rightarrow \quad \underbrace{u^{(s+1)}}_{=0} = z^{s+1} u + K(z^s T + z^{s-1} T' + \dots + T^{(s)})$$

(since u poly. of degree s)

$$\Rightarrow \quad u = -\frac{K}{z^{s+1}} (z^s T + z^{s-1} T' + \dots + T^{(s)})$$

Since $R(z) = \frac{u(1)}{u(0)}$, the result follows.

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We have $R(z) \approx e^z$, but how close?

<u>Lemma 1:</u> Let $M(t)$ be any polynomial of degree s s.t. $M^{(s)}(t) = 1$. (normalising)

Define
$$R(z) = \frac{M^{(s)}(1) + M^{(s-1)}(1) z + \dots + M'(1) z^{s-1} + M(1) z^s}{M^{(s)}(0) + M^{(s-1)}(0) z + \dots + M'(0) z^{s-1} + M(0) z^s}$$

$$\Rightarrow \quad e^z - R(z) = \mathcal{O}(z^{s+1}) \quad \text{for } z \to 0.$$

<u>Proof:</u> (formal)

Define $v(t) = M^{(s)}(t) + \dots + z^s M(t)$

Then
$$v'(t) = z M^{(s)}(t) + \dots + z^s M'(t)$$
$$= z\, v(t) - z^{s+1} M(t).$$

solving the ODE

$$\Rightarrow \quad v(t) = e^{zt} v(0) - z^{s+1} \int_0^t e^{z(t-r)} M(r)\, dr$$

evaluating $t=1$

$$\Rightarrow \quad e^z - \underbrace{\frac{v(1)}{v(0)}}_{= R(z)} = \frac{z^{s+1}}{v(0)} \int_0^1 e^{z(1-r)} M(r)\, dr$$

For $z \to 0$, it holds

- $e^{z(1-r)} = 1 + z(1-r) + \dots = 1 + \mathcal{O}(z)$
- $V(0) = 1 + z M^{(s-1)}(0) + \dots = 1 + \mathcal{O}(z)$

$\uparrow M^{(s)}(t) = 1$ (assumption)

$$\Rightarrow \frac{\int_0^1 e^{z(1-r)} M(r)\, dr}{V(0)} = \mathcal{O}(1+z)$$

polynomial, bounded on $]0,1[$

$$\Rightarrow e^z - R(z) = z^{s+1} \int_0^1 M(r)\, dr + \mathcal{O}(z^{s+2})$$
$$= \mathcal{O}(z^{s+1}) \quad .$$

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Rem: This implies that for a collocation method
$$R(z) - e^z = \mathcal{O}(z^{s+1})$$

Lemma 2: Let $R(z) = \frac{P(z)}{Q(z)}$, P, Q polynomials of degree $\leq s$.

Assume $Q(0) = 1$ and that $e^z - R(z) = \mathcal{O}(z^{s+1})$.

Then there exists a polynomial $M(t)$ s.t. $M^{(s)}(t) = 1$ and $\deg(M) = s$ s.t.

$$R(z) = \frac{M^{(s)}(1) + z M^{(s-1)}(1) + \dots + M(1) z^s}{M^{(s)}(0) + z M^{(s-1)}(0) + \dots + M(0) z^s}$$

Proof: Consider $Q(z) = 1 + q_1 z + q_2 z^2 + \dots + q_s z^s$ and define

$$M(t) = q_s + q_{s-1} t + q_{s-2} \frac{t^2}{2!} + \dots + q_1 \frac{t^{s-1}}{(s-1)!} + \frac{t^s}{s!}$$

$\Rightarrow M^{(k)}(0) = q_{s-k}$

$\Rightarrow Q(z) = M^{(s)}(0) + z M^{(s-1)}(0) + \dots + z^s M(0)$.

$\Rightarrow$ denominator okay. (by construction)

Next, let

$$\tilde{P}(z) = M^{(s)}(1) + M^{(s-1)}(1) z + \cdots + M(1) z^s .$$

By lemma 1, $\quad e^z - \frac{\tilde{P}(z)}{Q(z)} = \mathcal{O}(z^{s+1})$

By assumption, $\quad e^z - \frac{P(z)}{Q(z)} = \mathcal{O}(z^{s+1})$

$\Rightarrow \quad \frac{\tilde{P}(z)}{Q(z)} - \frac{P(z)}{Q(z)} = \mathcal{O}(z^{s+1})$

$\Rightarrow \quad \tilde{P}(z) - P(z) = \mathcal{O}(z^{s+1})$, since $Q(z) = 1 + \mathcal{O}(z)$

but $\deg(\tilde{P}) \leq s$, $\deg(P) \leq s$

$\Rightarrow \quad P = \tilde{P}$.

$\#$

Padé approximations of e^z

Goal: Fixed k, j find $R(z) = \frac{P(z)}{Q(z)}$ with $\deg(P) = k$, $\deg(Q) = j$ and s.t. $R(z) \approx e^z$ as good as possible.

Theorem: let $k, j \in \mathbb{N}$ be given. There exists an unique rational function

$$R_{kj}(z) = \frac{P_{kj}(z)}{Q_{kj}(z)} \quad \text{s.t. } \deg(P_{kj}) = k, \ \deg(Q_{kj}) = j$$

and s.t. $e^z - R_{kj}(z) = \mathcal{O}(z^{j+k+1})$,

i.e. R_{kj} is an approximation of the exp. of order $s = j+k$.

In particular,

$$P_{kj}(z) = 1 + \sum_{i=1}^{k} \frac{k!}{(k-i)!} \frac{(j+k-i)!}{(j+k)!} \frac{z^i}{i!}$$

$$Q_{kj}(z) = 1 + \sum_{i=1}^{j} (-1)^i \frac{j!}{(j-i)!} \frac{(j+k-i)!}{(j+k)!} \frac{z^i}{i!} = P_{jk}(-z)$$

<u>Proof:</u> Consider $M(t)$ polynom of degree $s=j+k$ s.t. $M^{(s)}(t)=1$, and

$$R(z)=\frac{M^{(s)}(1)+zM^{(s-1)}(1)+\dots+z^sM(1)}{M^{(s)}(0)+zM^{(s-1)}(0)+\dots+z^sM(0)}=\frac{P(z)}{Q(z)}$$

By lemma 1 : $R(z)-e^z=\mathcal{O}(z^{s+1})$

We want $\deg(P)=k$, $\deg(Q)=j$, so

$$(*)\begin{cases} M(1)=M'(1)=\dots=M^{(\overbrace{s-k-1}^{=j-1})}(1)=0 \Rightarrow \deg(P)=k \\ M(0)=M'(0)=\dots=M^{(\overbrace{s-j-1}^{=k-1})}(0)=0 \Rightarrow \deg(Q)=j \end{cases}$$

If we choose $M(t)=C\,t^k(t-1)^j$,

then the conditions above are satisfied, and since $M^{(s)}(t)=1$,

then $M(t)=\frac{t^k(t-1)^j}{(k+j)!}$

This proves existence.

For <u>uniqueness</u>, assume $\tilde{R}(z)=\frac{\tilde{P}(z)}{\tilde{Q}(z)}$, $\deg(\tilde{P})=k$, $\deg(\tilde{Q})=j$

and $\tilde{R}(z)-e^z=\mathcal{O}(z^{j+k+1})$

$\overset{\text{lemma 2}}{\Rightarrow}$ $\exists\ \tilde{M}(t)$ of degree $j+k$ s.t. $\tilde{M}^{(s)}(t)=1$

and s.t.

$$\tilde{R}(z)=\frac{\tilde{M}^{(s)}(1)+z\tilde{M}^{(s-1)}(1)+\dots+z^s\tilde{M}(1)}{\tilde{M}^{(s)}(0)+z\tilde{M}^{(s-1)}(0)+\dots+z^s\tilde{M}(0)}$$

As $\deg(\tilde{P})=k$, $\deg(\tilde{Q})=j$, so $\tilde{M}$ has to satisfy the conditions $(*)$

$$\Rightarrow \tilde{M}(t)=\frac{t^k(t-1)^j}{(j+k)!}$$

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Corollary: Stability function of

Gauss is $R_{s,s}(z)$
Radau is $R_{s-1,s}(z)$
Lobatto is $R_{s-1,s-1}(z)$.

Proof: Exercises.

Examples Padé rational approximations of the exponential for $k,j = 0,1,2$

	$k=0$	$k=1$	$k=2$
$j=0$	1	$1+z$	$1+z+\frac{1}{2}z^2$
$j=1$	$\frac{1}{1-z}$	$\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$	$\frac{1+\frac{2}{3}z+\frac{1}{6}z^2}{1-\frac{1}{3}z}$
$j=2$	$\frac{1}{1-z+\frac{1}{2}z^2}$	$\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\frac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$

Rem: (i) Stability function of Gauss and Lobatto IIIA on the diagonal of the table.

(ii) Stability function of Radau on the lower diagonal.

(see also exercises).

Order stars

Given $R_{k,j}(z)$, we want to find the corresponding stability domain

$$S = \{\, |R_{k,j}(z)| \leq 1 \}.$$

In 1978, Wanner and Hairer and Nørsett found that it is easier to consider the <u>order star</u>

$$A = \{ z \in \mathbb{C} \mid |R(z)| > |e^{z}| \}$$
$$= \{ z \in \mathbb{C} \mid |q(z)| > 1 \} \quad \text{with } q = \frac{R}{e^{z}}.$$

This is mostly relevant from a theoretical point of view

(For details, see Wanner, Hairer & Nørsett, 1978, Order stars and stability theorems).