Numerical Approximation of PDEs

Spring Semester 2025

Lecturer: Prof. Annalisa Buffa Assistant: Mohamed Ben Abdelouahab

Session 6: April 3, 2025

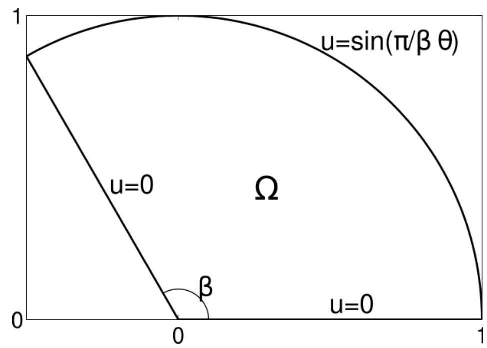

Figure 1: Domain and boundary conditions for exercise 1.

Exercise 1. Consider the equation $-\Delta u = 0$, with domain Ω and boundary conditions as in Figure 1.

1. Compute $\alpha \in \mathbb{R}^+$ such that $u = \rho^\alpha \sin\left(\frac{\pi}{\beta}\theta\right)$ is a solution of the problem.
2. Determine a condition on $\beta \in (0, 2\pi)$ such that $u \in H^1(\Omega)$ and a condition on $\beta \in (0, 2\pi)$ such that $u \in H^2(\Omega)$.
3. Complete the provided template `code_06_01_template.py` to perform a refinement study of the FEM approximation of the problem for $\beta = \pi/2$ and $\beta = 3\pi/2$. Check the convergence orders. What do you conclude ?

Hint: switch to polar coordinates, compute the analytic form of u, and then check the integrability of both $(\partial_x u)^2 + (\partial_y u)^2$ and $(\partial_{xx} u)^2 + 2(\partial_{xy} u)^2 + (\partial_{yy} u)^2$. Recall that polar coordinates read

$$\begin{cases} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{cases} \quad i.e. \quad \begin{cases} \rho &= \sqrt{x^2 + y^2}\,, \\ \theta &= \arctan\left(\frac{y}{x}\right)\,. \end{cases}$$

Moreover, the following identities hold:

$$\begin{aligned} \Delta u &= \frac{1}{\rho}\partial_\rho u + \partial_{\rho\rho} u + \frac{1}{\rho^2}\partial_{\theta\theta} u\,, \\ |\nabla u|^2 &= (\partial_\rho u)^2 + \frac{1}{\rho^2}(\partial_\theta u)^2\,, \\ (\partial_{xx} u)^2 + 2(\partial_{xy} u)^2 + (\partial_{yy} u)^2 &= (\partial_{\rho\rho} u)^2 + 2\left(\partial_\rho\left(\frac{1}{\rho}\partial_\theta u\right)\right)^2 + \left(\frac{1}{\rho^2}\partial_{\theta\theta} u + \frac{1}{\rho}\partial_\rho u\right)^2. \end{aligned}$$

Note that, for the sake of readability, here we simply write $\partial_x u$ instead of the more complete $\partial_x u(\rho(x,y),\theta(x,y))\big|_{\rho,\theta}$, and so on. Recall also that here we have

$$\int_\Omega f(x,y)\,dxdy = \int_0^1 \int_0^\beta f(x(\rho,\theta),y(\rho,\theta))\rho\,d\theta d\rho\,.$$

Solution:

1. *Note that $u = \rho^\alpha \sin\left(\frac{\pi}{\beta}\theta\right)$ fulfils the boundary conditions for every $\alpha \in \mathbb{R}^+$. Thus, we only need to enforce $\Delta u = 0$. For the Laplacian of u, we compute that*

$$\begin{aligned}\Delta u &= \frac{1}{\rho}\alpha\rho^{\alpha-1}\sin\left(\frac{\pi}{\beta}\theta\right) + \alpha(\alpha-1)\rho^{\alpha-2}\sin\left(\frac{\pi}{\beta}\theta\right) - \frac{1}{\rho^2}\rho^\alpha\frac{\pi^2}{\beta^2}\sin\left(\frac{\pi}{\beta}\theta\right)\\ &= \left(\alpha + \alpha(\alpha-1) - \frac{\pi^2}{\beta^2}\right)\rho^{\alpha-2}\sin\left(\frac{\pi}{\beta}\theta\right)\,.\end{aligned}$$

Consequently, u solves the PDE in Ω if and only if $\alpha = \frac{\pi}{\beta}$. In that case the solution reads $u = \rho^{\frac{\pi}{\beta}}\sin\left(\frac{\pi}{\beta}\theta\right) = \rho^\alpha\sin(\alpha\theta)$.

2. *It holds that:*
   - $\partial_\rho u = \alpha\rho^{(\alpha-1)}\sin(\alpha\theta)$
   - $\partial_{\rho\rho} u = \alpha(\alpha-1)\rho^{(\alpha-2)}\sin(\alpha\theta)$
   - $\partial_\theta u = \rho^\alpha\alpha\cos(\alpha\theta)$
   - $\partial_{\theta\theta} u = -\rho^\alpha\alpha^2\sin(\alpha\theta)$
   - $\partial_{\rho\theta} u = \alpha^2\rho^{(\alpha-1)}\cos(\alpha\theta)$

   First, we check that $u \in H^1(\Omega)$:

$$\begin{aligned}\int_\Omega |\nabla u|^2 dxdy &= \int_0^1\int_0^\beta \left((\partial_\rho u)^2 + \frac{1}{\rho^2}(\partial_\theta u)^2\right)\rho\,d\theta\,d\rho\\ &= \int_0^1\int_0^\beta\left(\alpha^2\rho^{2\alpha-2}\sin^2(\alpha\theta) + \frac{1}{\rho^2}\rho^{2\alpha}\alpha^2\cos^2(\alpha\theta)\right)\rho\,d\theta\,d\rho\\ &= \pi\alpha\int_0^1 \rho^{2\alpha-1}\,d\rho\,,\end{aligned}$$

   which is integrable, if $2\alpha - 1 > -1$, i.e. $\alpha = \frac{\pi}{\beta} > 0$. Thus $u \in H^1(\Omega)$ for every $\beta \in (0,2\pi)$ and $\|\nabla u\|^2_{L^2(\Omega)} = \frac{\pi}{2}$.

   Using the same arguments as above, we check the H^2 semi-norm and find

$$\begin{aligned}|u|^2_{H^2(\Omega)} &= \int_\Omega \left((\partial_{xx}u)^2 + 2(\partial_{xy}u)^2 + (\partial_{yy}u)^2\right)\,dx\,dy\\ &= \int_0^1\int_0^\beta\left((\partial_{\rho\rho}u)^2 + 2\big(\partial_\rho(\rho^{-1}\partial_\theta u)\big)^2 + \big(\rho^{-2}\partial_{\theta\theta}u + \rho^{-1}\partial_\rho u\big)^2\right)\rho\,d\theta\,d\rho\\ &= 2\pi\alpha(\alpha-1)^2\int_0^1 \rho^{2\alpha-3}\,d\rho\,.\end{aligned}$$

Consequently, the H^2 semi-norm is finite, if $\rho^{2\alpha-3}$ is integrable or if $\alpha = 1$. That is, $u \in H^2(\Omega)$, if

$$2\alpha - 3 > -1 \ \vee \ \alpha = 1 \quad \Leftrightarrow \quad \alpha = \frac{\pi}{\beta} \geq 1 \quad \Leftrightarrow \quad \beta \leq \pi \, .$$

In other words, the domain Ω has to be convex.

3. *The solution script is provided on Moodle.*

Exercise 2. Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain with a sequence of triangulations $\mathcal{T}_h$ indexed over $h > 0$. The sequence of triangulations is shape-regular and quasi-uniform. Suppose that the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a weak solution $u \in H^2(\Omega)$ for any $f \in L^2(\Omega)$ and that

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{2}$$

Let u_h be the Galerkin solution using piecewise linear finite elements. Show that for any $g \in L^2(\Omega)$, we have the convergence estimate

$$\left| \int_\Omega g(u - u_h) \right| \leq Ch^2 \|g\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}.$$

You can use a technique similar as in the proof of the Aubin-Nitsche lemma.
Lastly, interpret the result in the case $g = 1$.

Solution:
Proof 1: we use the Aubin-Nitsche lemma and estimate

$$\begin{aligned} \left| \int_\Omega g(u - u_h) \right| &\leq \|g\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)} \\ &\leq \|g\|_{L^2(\Omega)} Ch^2 \|u\|_{H^2(\Omega)} \leq Ch^2 \|g\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}. \end{aligned}$$

Proof 2: We let $z \in H^2(\Omega)$ be the unique weak solution of

$$\begin{aligned} -\Delta z &= g \quad \textit{in } \Omega \\ z &= 0 \quad \textit{on } \partial\Omega. \end{aligned} \tag{3}$$

Let z_h be the finite element approximation to that problem. Then we observe

$$\begin{aligned} \int_\Omega g(u - u_h)dx &= \int_\Omega \nabla z \nabla (u - u_h) dx \\ &= \int_\Omega \nabla (z - z_h) \nabla (u - u_h) dx. \end{aligned}$$

Hence

$$\left| \int_\Omega g(u - u_h) \right| \leq \|\nabla (u - u_h)\|_{L^2(\Omega)} \|\nabla (z - z_h)\|_{L^2(\Omega)}.$$

The proof now follows with two estimates

$$\begin{aligned}\|\nabla(u-u_h)\|_{L^2(\Omega)} &\le Ch\|u\|_{H^2(\Omega)} \le Ch\|f\|_{L^2(\Omega)},\\ \|\nabla(z-z_h)\|_{L^2(\Omega)} &\le Ch\|z\|_{H^2(\Omega)} \le Ch\|g\|_{L^2(\Omega)}.\end{aligned}$$

This completes the proof of the estimate.
In the case that $g=1$, this tells us that the average converges faster than the H^1 error.

Exercise 3. Let Ω be a domain in $\mathbb{R}^2$ and consider diffusion-convection-reaction problem:

$$\begin{aligned}-\epsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f \text{ over } \Omega,\\ u &= 0 \text{ along } \Gamma_D,\\ \nabla u\cdot\mathbf{n} &= 0 \text{ along } \Gamma_N\end{aligned}$$

where we use the boundary partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ into a Dirichlet and Neumann boundary part, $\Gamma_D \cap \Gamma_N = \emptyset$. Here, we have used the outward pointing unit normal $\mathbf{n}$.
We assume that

$$\begin{aligned}c - \frac{1}{2}\operatorname{div} b &\ge 0,\\ \mathbf{b}\cdot\mathbf{n} \ge 0 &\text{ along } \Gamma_N.\end{aligned}$$

State the weak formulation of this problem. Find the continuity and coercivity constants of the bilinear form.

Solution:
The weak formulation is:

$$a(u,v) = \int_\Omega \epsilon\nabla u\nabla v + \mathbf{b}\nabla u\cdot v + cuv = \int_\Omega fv.$$

We estimate the continuity constant in the usual manner:

$$\begin{aligned}|a(u,v)| &\le \int_\Omega \epsilon|\nabla u|\cdot|\nabla v| + |\mathbf{b}||\nabla u|\cdot|v| + |c||u||v|\\ &\le \left(\epsilon + \|\mathbf{b}\|_\infty + \|c\|_\infty\right)\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

We estimate the coercivity constant as follows.

$$a(u,u) = \int_\Omega \epsilon|\nabla u|^2 + \mathbf{b}\nabla u\cdot u + cu^2 = \int_\Omega \epsilon|\nabla u|^2 + cu^2 + \int_\Omega \mathbf{b}\nabla u\cdot u.$$

Now we find that

$$\int_\Omega \mathbf{b}\nabla u\cdot u = \frac{1}{2}\int_\Omega \mathbf{b}\nabla(u^2) = \frac{1}{2}\int_\Omega \operatorname{div}\left(\mathbf{b}u^2\right) - \frac{1}{2}\int_\Omega \operatorname{div}\mathbf{b}\cdot u^2.$$

We use the divergence theorem, together with boundary conditions along Γ_D and the outflow condition along Γ_N:

$$\int_\Omega \operatorname{div}\left(\mathbf{b}u^2\right) = \int_{\partial\Omega}\mathbf{b}\cdot\mathbf{n}u^2 = \underbrace{\int_{\Gamma_D}\mathbf{b}\cdot\mathbf{n}\cdot u^2}_{u_{|\Gamma_D}=0} + \underbrace{\int_{\Gamma_N}\mathbf{b}\cdot\mathbf{n}\cdot u^2}_{\mathbf{b}\cdot\mathbf{n}\ge 0} \ge 0.$$

Consequently,

$$a(u,u) \geq \int_\Omega \epsilon|\nabla u|^2 + \left(c - \frac{1}{2}\operatorname{div}\mathbf{b}\right)u^2 \geq \int_\Omega \epsilon|\nabla u|^2.$$

We thus find

$$a(u,u) \geq \frac{\epsilon}{1+C_F^2}\|u\|_{H^1}.$$

This shows the desired estimates.

Exercise 4. The goal of this exercise is to prove a discrete maximum principle for $\mathbb{P}_1$ finite elements in two dimensions $d = 2$.

1. A real square matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is called an M-matrix if the following is true:
   - The diagonal elements are positive: $a_{ii} > 0$ for all i.
   - The sum of elements in each row is positive: $\sum_{k=1}^n a_{ik} > 0$ for all i.
   - The off-diagonal elements are non-positive: $a_{ij} \leq 0$ for all $i \neq j$.

   Show that A is invertible and that all the coefficients of its inverse are non-negative.

2. Consider the numerical solution u_h of the Poisson-Dirichlet problem (1) using $\mathbb{P}_1$ finite elements method on a triangulation mesh where all triangle angles are at most $\pi/2$. Show that if $f \geq 0$ then $u_h \geq 0$ in Ω.

Hint: For *1*, consider a pair of vectors (x, y) in $\mathbb{R}^n$ such that $Ax = y$ and $y \geq 0$ (meaning that all the components of the vector y are non-negative), prove that $x \geq 0$ and conclude that A is injective. For *2*, consider the stiffness matrix A_h associated with this system and show that for every $\varepsilon > 0$, the matrix $A_h + \varepsilon I$ is an M-matrix, and consequently, A_h^{-1} has non-negative elements.

Solution:

1. *Let A be an M-matrix and consider a vector $x \in \mathbb{R}^n$ such that $Ax = y \geq 0$. Define the index i_0 as*

$$x_{i_0} = \min_{1\leq i\leq n} x_i. \tag{4}$$

we can write:

$$a_{i_0 i_0}x_{i_0} + \sum_{j\neq i_0} a_{i_0 j}x_j = y_{i_0} \geq 0. \tag{5}$$

Rearranging this equation, we obtain:

$$\left(\sum_{j=1}^n a_{i_0 j}\right)x_{i_0} \geq \sum_{j\neq i_0} a_{i_0 j}(x_{i_0} - x_j). \tag{6}$$

By the definition of i_0, we have $x_{i_0} \leq x_j$ for all j, and since the off-diagonal elements satisfy $a_{i_0 j} \leq 0$, it follows that:

$$x_{i_0} \geq 0. \tag{7}$$

Thus, since x_{i_0} is the smallest component of x, we conclude that $x \geq 0$.

Now, suppose for some $x \in \mathbb{R}^n$ we have $Ax = 0$. This implies that $x = 0$ since $x \geq 0$ and $-x \geq 0$. We deduce that A is invertible because injective. Furthermore, since $Ax = y \geq 0$ implies $x \geq 0$ and $x = A^{-1}y$, we can take y as an arbitrary vector from the canonical basis of $\mathbb{R}^n$, and we obtain $A^{-1}_{ij} = x_i \geq 0$ for all $1 \leq i, j \leq n$.

2. *First, the diagonal elements of A_h are positive:*

$$(A_h)_{ii} = \int_\Omega |\nabla \varphi_i|^2 > 0. \tag{8}$$

Consider two distinct nodes v_i and v_j sharing a common triangle K in the mesh. The basis function φ_i has trace zero on the edge opposite to the vertex v_i of K, same holds for φ_j. It follows that the gradients $\nabla\varphi_i$ and $\nabla\varphi_j$ are orthogonal to the corresponding opposite edge to each vertex.

Now, let α be the angle formed by $\nabla\varphi_i$ and $\nabla\varphi_j$, and let β be the angle at the third vertex of K, other than v_i and v_j. We have then $\beta = \pi - \alpha$. Since we assume that all triangle angles are at most $\pi/2$, then $\beta \geq \pi/2$, implying:

$$\nabla\varphi_i \cdot \nabla\varphi_j \leq 0. \tag{9}$$

Integrating over the domain Ω, we obtain:

$$(A_h)_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j, dx \leq 0, \quad \forall i \neq j. \tag{10}$$

Let N the total number of nodes and N_0 be the number of interior nodes, so that we have the nodes $\{v_i\}_{N_0 < i \leq N}$ at the boundary $\partial\Omega$ and the matrix A_h is of shape $N_0 \times N_0$. Using the partition of unity property of $\mathbb{P}_1$ finite elements basis:

$$1 = \sum_{j=1}^{N} \varphi_j, \tag{11}$$

and take the gradient for every $1 \leq i \leq N_0$:

$$\sum_{j=1}^{N_0} \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j, dx = - \sum_{j=N_0+1}^{N} \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j, dx. \tag{12}$$

using (10), we deduce:

$$\sum_{i=1}^{N_0} (A_h)_{ij} \geq 0. \tag{13}$$

From properties (8), (10) and (13), it follows that $A_h + \varepsilon I$ is an M-matrix for some $\varepsilon > 0$ and $(A_h + \varepsilon I)^{-1}$ has non-negative entries according to question 1. The inverse application being continuous on the set of invertible matrices, we deduce by taking the limit $\varepsilon \to 0$ that A_h^{-1} has also non-negative entries which concludes the proof.