Numerical Approximation of PDEs

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Exercise 1. [Streamline Diffusion for Diffusion-Advection-Reaction Equation] Let Ω be a bounded domain in $\mathbb{R}^2$. We consider the following advection-diffusion-reaction problem for $u : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\Delta u + \operatorname{div}(\boldsymbol{\beta} u) + u &= f && \text{in } \Omega \\ u &= \phi && \text{on } \Gamma_D \ , \\ \nabla u \cdot \mathbf{n} &= u \boldsymbol{\beta} \cdot \mathbf{n} && \text{on } \Gamma_N \end{aligned} \tag{1}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ is a partition of the boundary and where and $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x}) \in \mathbb{R}^2$ is the flow vector field and $f \in L^2(\Omega)$ the source term.

1. Derive the weak formulation of the problem; state the bilinear form a and the functional on the right-hand side. Apply integration by parts to the first-order term and use the boundary conditions to remove all boundary integrals.

2. For now we take $f = 0$.
   We can use the bilinear form $a(\,\cdot\,,\,\cdot\,)$ for a finite element method, similar as we have done for the Poisson problem. But in the case of an advection-dominated problem, we need to stabilize the numerical method to avoid spurious oscillations in the numerical solution. Introduce an isotropic *streamline viscosity* by modifying the bilinear form as follows:

   $$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{K \in \mathcal{T}_h} \frac{\gamma h_K}{\|\boldsymbol{\beta}\|_{L^\infty(K)}} \int_K (\nabla u_h \cdot \boldsymbol{\beta})\,(\nabla v_h \cdot \boldsymbol{\beta}) \,,$$

   where $\gamma > 0$ is a stabilisation parameter. We mention that the use of the $\|\cdot\|_\infty$ norm is equivalent to the norm discussed in class (only potentially requiring a differing value of γ) but easier to implement.
   Now we assume that $\Omega = [0,1]^2$, $\boldsymbol{\beta} = (-10^3, -10^3)^T$ and $\Gamma_D = \partial\Omega$. We also define ϕ as:

   $$\begin{cases} \phi = 1 & \text{on } \Gamma = \{(x,y) \mid 0 \le x \le 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } 0 \le y \le 1\} \\ \phi = 0 & \text{on } \Gamma_D \setminus \Gamma. \end{cases}$$

   Construct $a_h(u_h, v_h)$ with your Python code by adding the aforementioned contribution

   $$+ \sum_{K \in \mathcal{T}_h} \frac{\gamma h_K}{\|\boldsymbol{\beta}\|_{L^\infty(K)}} \int_K (\nabla u_h \cdot \boldsymbol{\beta})\,(\nabla v_h \cdot \boldsymbol{\beta})$$

   to the bilinear form. Solve the problem without stabilisation for $h \in \{0.1, 0.05, 0.025\}$ without stabilisation and then with streamline diffusion stabilisation for all values of

h and $\gamma \in \{0.1, 1, 5\}$. In the Python template `conv_dom_template.py` you will find a function for solving without stabilisation and with streamline diffusion stabilisation. In `integrate_template.py`, you will find templates for iterators constructing the transport matrix $B_{ij} = -\int_\Omega \phi_i \boldsymbol{\beta} \cdot \nabla \phi_j$ and a template for the streamline diffusion stabilisation as discussed above. After you have completed the script, rename it to `integrate.py` to run the main script.

Compare the stabilised and non-stabilised solutions. Is the stabilisation consistent for $f = 0$? And what about $f \neq 0$?

Exercise 2. [The plate problem and the Argyris finite element] The goal of this exercise is to solve a simple elliptic fourth order PDE using a suitable (conforming) finite element method.

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Consider the biharmonic equation

$$\begin{cases} \Delta(\Delta u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

   Show that we have the following variational formulation, and use Lax-Milgram lemma to prove the well-posedness of the weak problem.

$$\int_\Omega \Delta u \, \Delta w \, dx = \int_\Omega f w \, dx \quad \text{for all } w \in H_0^2(\Omega). \tag{3}$$

   where $H_0^2(\Omega) = \left\{ u \in H^2(\Omega), \,\middle|\, u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$.

   Hint: Note that if $u \in H_0^2(\Omega)$, then we have $\dfrac{\partial u}{\partial x_i} \in H_0^1(\Omega)$, $i = 1, \ldots, n$ and by integrating by parts twice:

$$\int_\Omega |\Delta u|^2 \, dx = \sum_{i,j=1}^n \int_\Omega \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx.$$

2. Let $\mathcal{T}^h$ be a conforming triangulation of $\overline{\Omega} \subset \mathbb{R}^2$ into elements $\{K_j\}$. Let $\mathcal{P}_{K_j} \subset H^2(K_j)$ be a polynomial space on each element. Define the global finite element space $V_h \subset C^1(\overline{\Omega})$ such that $v_h|_{K_j} \in \mathcal{P}_{K_j}$ for all j. Define:

$$V_h^0 = \{v_h \in V_h : v_h = 0 \text{ on } \partial\Omega\},$$
$$V_h^{00} = \left\{ v_h \in V_h : v_h = 0 \text{ and } \frac{\partial v_h}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

   Show that:

$$V_h^0 \subset H^2(\Omega) \cap H_0^1(\Omega), \text{ and } \quad V_h^{00} \subset H_0^2(\Omega).$$

3. Denote by $\mathbb{P}_5(K)$ the space of polynomials of total degree at most 5 on the triangle K. Consider the following set of degrees of freedom

$$\mathcal{A} = \left\{ p(v_k), \quad \frac{\partial p}{\partial x_i}(v_k), \quad \frac{\partial^2 p}{\partial x_i \partial x_j}(v_k), \quad \frac{\partial p}{\partial n}(m_k), \qquad k = 1, 2, 3, \quad i, j = 1, 2 \right\}, \tag{4}$$

where v_1, v_2, v_3 are the triangle's vertices and m_1, m_2, m_3 are the midpoints of its edges. $\frac{\partial p}{\partial n}(m_k)$ denotes the normal derivative of p along the edge at midpoint m_k.

Show that $\mathcal{A}$ (called the Argyris element) is $\mathbb{P}_5$-unisolvent i.e. any polynomial $p \in \mathbb{P}_5$ is uniquely determined on K by the 21 degrees of freedom in (4).

4. We are now ready to construct a discrete conforming subspace and solve the biharmonic equation (2) numerically in two dimensions $n = 2$ based on the variational formulation (3). Consider the discrete space V_h associated to the Argyris element. Show that V_h is a subspace of $H^2(\Omega)$. We denote

$$V_h = \left\{ v \in C^1(\overline{\Omega}) \,\middle|\, v|_K \in \mathbb{P}_5 \text{ for each } K \in \mathcal{T}_h \right\}.$$

Hint: Show that for two adjacent triangles K_1 and K_2 sharing a common edge $\Gamma = [v_1, v_2]$. We have $v \in C^1(K_1 \cup K_2) \cap V_h$ if and only if for $k = 1, 2$ the following conditions are satisfied:

$$\begin{aligned} p|_{K_1}(v_k) &= p|_{K_2}(v_k), \\ \frac{\partial(p|_{K_1})}{\partial x_i}(v_k) &= \frac{\partial(p|_{K_2})}{\partial x_i}(v_k), \\ \frac{\partial^2(p|_{K_1})}{\partial x_i \partial x_j}(v_k) &= \frac{\partial^2(p|_{K_2})}{\partial x_i \partial x_j}(v_k), \\ \frac{\partial(p|_{K_1})}{\partial n}(m) &= \frac{\partial(p|_{K_2})}{\partial n}(m), \end{aligned}$$

where $i, j = 1, 2$ and m is the midpoint of the shared edge. Use question 2 to conclude.

5. Suppose that the regularity of the exact solution of (3) is $u \in H^4(\Omega) \bigcap_{K \in \mathcal{T}_h} H^6(K)$, and that the interpolation operator $\Pi_K : C^2(\Omega) \to \mathbb{P}_5(K)$ associated with $\mathcal{A}$ satisfies the following estimate:

$$\forall K \in \mathcal{T}_h, \quad \|v - \Pi_K v\|_{H^2(K)} \lesssim h_K^4 |v|_{H^6(K)}, \quad \forall v \in H^6(K), \text{ where } h_K = \operatorname{diam}(K). \tag{5}$$

Let $u_h \in V_h$ be the approximate solution of (3). Show that the global error satisfies the estimate:

$$\|u - u_h\|_{H^2(\Omega)} \lesssim h^2 |u|_{H^4(\Omega)}, \quad \text{where } h = \max_{K \in \mathcal{T}_h} h_K. \tag{6}$$