Numerical Approximation of PDEs

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Lecturer: Prof. Annalisa Buffa   Assistant: Mohamed Ben Abdelouahab

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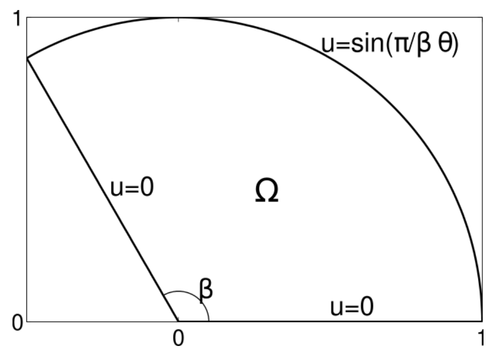

Figure 1: Domain and boundary conditions for exercise 1.

Exercise 1. Consider the equation $-\Delta u = 0$, with domain Ω and boundary conditions as in Figure 1.

1. Compute $\alpha \in \mathbb{R}^+$ such that $u = \rho^\alpha \sin\left(\frac{\pi}{\beta}\theta\right)$ is a solution of the problem.
2. Determine a condition on $\beta \in (0, 2\pi)$ such that $u \in H^1(\Omega)$ and a condition on $\beta \in (0, 2\pi)$ such that $u \in H^2(\Omega)$.
3. Complete the provided template `code_06_01_template.py` to perform a refinement study of the FEM approximation of the problem for $\beta = \pi/2$ and $\beta = 3\pi/2$. Check the convergence orders. What do you conclude ?

Hint: switch to polar coordinates, compute the analytic form of u, and then check the integrability of both $(\partial_x u)^2 + (\partial_y u)^2$ and $(\partial_{xx} u)^2 + 2(\partial_{xy} u)^2 + (\partial_{yy} u)^2$. Recall that polar coordinates read

$$\begin{cases} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{cases} \quad i.e. \quad \begin{cases} \rho &= \sqrt{x^2+y^2}\,, \\ \theta &= \arctan\left(\frac{y}{x}\right)\,. \end{cases}$$

Moreover, the following identities hold:

$$\begin{aligned} \Delta u &= \frac{1}{\rho}\partial_\rho u + \partial_{\rho\rho} u + \frac{1}{\rho^2}\partial_{\theta\theta} u\,, \\ |\nabla u|^2 &= (\partial_\rho u)^2 + \frac{1}{\rho^2}(\partial_\theta u)^2\,, \\ (\partial_{xx} u)^2 + 2(\partial_{xy} u)^2 + (\partial_{yy} u)^2 &= (\partial_{\rho\rho} u)^2 + 2\left(\partial_\rho\left(\frac{1}{\rho}\partial_\theta u\right)\right)^2 + \left(\frac{1}{\rho^2}\partial_{\theta\theta} u + \frac{1}{\rho}\partial_\rho u\right)^2. \end{aligned}$$

Note that, for the sake of readability, here we simply write $\partial_x u$ instead of the more complete $\partial_x u(\rho(x,y),\theta(x,y))\big|_{\rho,\theta}$, and so on. Recall also that here we have

$$\int_\Omega f(x,y)\,dxdy = \int_0^1 \int_0^\beta f(x(\rho,\theta),y(\rho,\theta))\rho\,d\theta d\rho\ .$$

Exercise 2. Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain with a sequence of triangulations $\mathcal{T}_h$ indexed over $h > 0$. The sequence of triangulations is shape-regular and quasi-uniform. Suppose that the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a weak solution $u \in H^2(\Omega)$ for any $f \in L^2(\Omega)$ and that

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{2}$$

Let u_h be the Galerkin solution using piecewise linear finite elements. Show that for any $g \in L^2(\Omega)$, we have the convergence estimate

$$\left|\int_\Omega g(u-u_h)\right| \leq Ch^2\|g\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}.$$

You can use a technique similar as in the proof of the Aubin-Nitsche lemma.
Lastly, interpret the result in the case $g = 1$.

Exercise 3. Let Ω be a domain in $\mathbb{R}^2$ and consider diffusion-convection-reaction problem:

$$\begin{aligned} -\epsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f \text{ over } \Omega, \\ u &= 0 \text{ along } \Gamma_D, \\ \nabla u \cdot \mathbf{n} &= 0 \text{ along } \Gamma_N \end{aligned}$$

where we use the boundary partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ into a Dirichlet and Neumann boundary part, $\Gamma_D \cap \Gamma_N = \emptyset$. Here, we have used the outward pointing unit normal $\mathbf{n}$.
We assume that

$$\begin{gathered} c - \frac{1}{2}\operatorname{div} b \geq 0, \\ \mathbf{b}\cdot\mathbf{n} \geq 0 \text{ along } \Gamma_N. \end{gathered}$$

State the weak formulation of this problem. Find the continuity and coercivity constants of the bilinear form.

Exercise 4. The goal of this exercise is to prove a discrete maximum principle for $\mathbb{P}_1$ finite elements in two dimensions $d = 2$.

1. A real square matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is called an M-matrix if the following is true:
   - The diagonal elements are positive: $a_{ii} > 0$ for all i.
   - The sum of elements in each row is positive: $\sum_{k=1}^n a_{ik} > 0$ for all i.

- The off-diagonal elements are non-positive: $a_{ij} \leq 0$ for all $i \neq j$.

Show that A is invertible and that all the coefficients of its inverse are non-negative.

2. Consider the numerical solution u_h of the Poisson-Dirichlet problem (1) using $\mathbb{P}_1$ finite elements method on a triangulation mesh where all triangle angles are at most $\pi/2$. Show that if $f \geq 0$ then $u_h \geq 0$ in Ω.

Hint: For *1*, consider a pair of vectors (x, y) in $\mathbb{R}^n$ such that $Ax = y$ and $y \geq 0$ (meaning that all the components of the vector y are non-negative), prove that $x \geq 0$ and conclude that A is injective. For *2*, consider the stiffness matrix A_h associated with this system and show that for every $\varepsilon > 0$, the matrix $A_h + \varepsilon I$ is an M-matrix, and consequently, A_h^{-1} has non-negative elements.