Numerical Approximation of PDEs

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Exercise 1. Let $I = [0, 1]$. Discuss whether the function

$$f : I \to \mathbb{R}, \quad x \mapsto x^{5/4}$$

is a member of $L^2(I)$ or $H^1(I)$. Is f' a member of $H^1(I)$?

Exercise 2. Consider the open half unit ball in $\mathbb{R}^2$, denoted by $B = \{x \in \mathbb{R}^2 : |x| < 1\} \cap \mathbb{R}^+$. Show that the function $u(x) = |\log(|x|/2)|^\lambda$ belongs to the space $H^1(B)$ for $0 < \lambda < \frac{1}{2}$, but is not bounded in any neighborhood of the origin.
Hint: Use the polar coordinate transformation to compute the integrals, and note that $\nabla|x| = \frac{x}{|x|}$ for any $x \neq 0$.

NB. *This is to see that in higher dimensions $d \geq 2$, a member of $H^1(\Omega)$ is not necessarily continuous. As a result, we can only refer to the value of a function $u \in H^1(\Omega)$ "almost everywhere" in Ω rather than in the usual pointwise sense. Because of this, it's not immediately clear how to define the "value at the boundary", or restriction of u on $\partial\Omega$, since $\partial\Omega$ is negligible set with respect to the d-dimensional Lebesgue measure. However, there is a way to define the trace $u|_{\partial\Omega}$ of an $H^1(\Omega)$ function. This result is formally established by the trace theorem.*

Exercise 3. [Poisson equation with mixed boundary conditions]
Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded Lipschitz domain whose boundary $\partial\Omega$ can be split into two (essentially) disjoint parts: $\partial\Omega = \Gamma_D \cup \Gamma_N$.
Let $f \in L^2(\Omega)$ and $g_N \in L^2(\Gamma_N)$. Let $g_D \in H^{1/2}(\Gamma_D)$, so that there exists $G \in H^1(\Omega)$ satisfying $\gamma|_{\Gamma_D}(G) = g_D$. In other words, the trace of G on Γ_D is g_D.
We consider the Poisson equation

$$\begin{aligned} -\Delta u(x) &= f(x) && \text{in } \Omega, \\ u(x) &= g_D(x) && \text{on } \Gamma_D, \\ \partial_n u(x) &= g_N(x) && \text{on } \Gamma_N, \end{aligned}$$

Define the sets

$$V_{g_D} = \left\{ v \in H^1(\Omega) : \gamma|_{\Gamma_D}(v) = g_D \right\},$$
$$V_0 = H^1_{0,\Gamma_D}(\Omega) = \left\{ v \in H^1(\Omega) : \gamma|_{\Gamma_D}(v) = 0 \right\}$$

1. Suppose that u solves the Poisson problem. Show that $u_0 = u - G$ belongs to V_0 and satisfies an equation of the form

$$a(u_0, v) = F(v), \quad \forall v \in V_0. \tag{1}$$

   Give the explicit expressions of a and F.

   Hint: Multiply by a test function in V_0 and perform integration by parts.

2. Show that the conditions of the Lax-Milgram lemma are satisfied and use it to show that Problem (1) is well-posed in V_0, i.e., there exists a unique $u_0 \in V_0$ such that

$$a(u_0, v) = F(v), \quad \forall v \in V_0.$$

Hint: The Poincaré inequality holds also in $V_0 : \forall v \in V_0, \|v\|_{L^2(\Omega)} \leq C_F \|\nabla v\|_{L^2(\Omega)}$.

3. Show there exists a unique $u \in V_{g_D}$ that is the solution of the original weak formulation.

4. Explain why we cannot apply the Lax-Milgram lemma directly to the set V_{g_D}.

Exercise 4. [Equivalence of hat functions with the space of piecewise linears]
Denote the discretization parameter by $h = \frac{1}{N}$, where $N \in \mathbb{N}^*$ and consider a uniform subdivision $\mathcal{E}_h$ of $[a, b]$: $\mathcal{E}_h = \{x_0 = a, x_1, x_2, \ldots, x_{N-1}, x_N = b\}$ such that $h = x_{i+1} - x_i$.

Consider the space

$$V_h := \left\{ v \in C^0(\Omega) : \quad v(a) = v(b) = 0 \quad \text{and} \quad v|_{I_i} \in \mathbb{P}_1 \right\},$$

where $I_i = [x_i, x_{i+1}]$ are the subintervals forming the partition of $[a, b]$ with N elements and $\mathbb{P}_1 = \{p \,|\, p(x) = ax + b,\ (a, b) \in \mathbb{R}\}$ is the space of linear polynomials.
Next, consider the space

$$W_h = \text{span}\left\{\lambda_1, \ldots, \lambda_{N-1}\right\},$$

where the λ_i are the hat functions (see figure 1) defined by:

$$\forall i \in [0 \ldots N] \ \forall x_j \in \mathcal{E}_h \quad \lambda_i \in V_h \quad \text{and} \quad \lambda_i(x_j) = \delta_{ij}.$$

Prove that $V_h = W_h$.

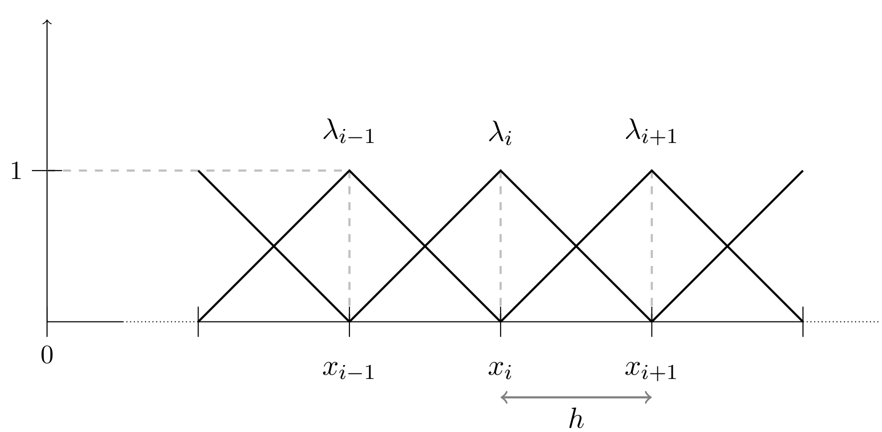

Figure 1: Basis hat functions λ_i.

Exercise 5. [Assembly of the mass matrix with nonconstant reaction term in $1D$]

Let $\Omega = (0, 1)$, we are considering the so-called mass matrix with a nonconstant reactivity $C^\infty(\Omega) \ni c(x) > 0$ which has the following entries:

$$M_{i,j} = \int_\Omega c(x)\lambda_i(x)\lambda_j(x)\mathrm{d}x. \tag{2}$$

The assembly iterates over all elements and then looks up which functions are nonzero on the element. The integral from (4) is split into its contributions from each element and then added to a sparse matrix at the right position.
Since $c(x)$ can be anything, the only way of computing the entries numerically is using a quadrature formula over the element (x_i, x_{i+1}). Use the provided script as a starting point to implement the assembly of this matrix for general $c(x)$ and then assemble M for $c(x) = 1 + \frac{1}{2}\sin(\pi x)$. The script provides a function for acquiring various Gauss quadrature formulas and for the midpoint rule.
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