EPFL
Mathematics section
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Numerical Approximation of Partial Differential Equations

MATH-451 EXAM **04.07.2023** **15h15-18h15**

Name: .... **Forename:** .... **Sciper:** ....

EXAM RULES:

- **CAMIPRO card is mandatory and will be checked.**
- **The exam is recorded only after the student has signed.**
- **Do not detach any page.**
- **Write with blue or black ink. No other colors are allowed.**
- **Mobile phones and other electronic devices must be turned off and in the bags.**
- **Please copy all Python code into the exam. Code on computers will not be graded.**
- **Justify all your answers. The clearness of the answers will be evaluated as well.**

☐ I read and understood the above rules. Signature : ....

Problems	Points	Grades
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

Problem 1 (10 points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary $\partial\Omega$ that can be split into three disjoint parts $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We consider an elliptic equation

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ \partial_n u &= g_2 \quad \text{on } \Gamma_2, \\ \partial_n u + ku &= g_3 \quad \text{on } \Gamma_3, \end{cases}$$

where $f \in L^2(\Omega)$, $g_2 \in L^2(\Gamma_2)$, $g_3 \in L^2(\Gamma_3)$ and $k \in \mathbb{R}$, $k > 0$.

(1) Write a weak formulation of the problem in suitable Sobolev spaces.

(2) Show that the problem is well-posed by using the Lax-Milgram theorem.

Problem 2 (10 points)

Let $\mathcal{T}_h$ be a regular affine triangulation of a convex, polygonal domain $\Omega \subset \mathbb{R}^2$.

(1) Consider the space X_h^1 of continuous $\mathbb{P}_1$ finite elements (continuous piecewise linear functions on $\mathcal{T}_h$).

(a) Define the space X_h^1 of finite elements of degree 1 on the mesh $\mathcal{T}_h$.

(b) Describe a possible set of nodal degrees of freedom and corresponding Lagrange basis functions and make a sketch of their shape.

(c) Construct explicitly the set of basis functions on the reference element $\hat{K} = \{\hat{x} \geq 0,\ \hat{y} \geq 0,\ \hat{x} + \hat{y} \leq 1\}$.

(2) Consider the space X_h^2 of continuous $\mathbb{P}_2$ finite elements (continuous piecewise quadratic functions on $\mathcal{T}_h$).

(a) Define the space X_h^2 of finite elements of degree 2 on the mesh $\mathcal{T}_h$.

(b) Construct explicitly a set of basis functions on the reference $\mathbb{P}_2$ element.

(c) Consider a function $u \in H^3(\Omega)$ and let $I_h^2 : C^0(\overline{\Omega}) \to X_h^2$ be the Lagrange interpolation operator. State precisely (without proof) the interpolation error estimates that are valid for

$$\|u - I_h^2 u\|_{L^2(\Omega)} \text{ and } \|u - I_h^2 u\|_{H^1(\Omega)}.$$

Problem 3 (10 points)

Consider a one-dimensional heat conduction problem described by the following heat equation:

$$\partial_t u = m \cdot \partial_x^2 u + f$$

over the interval $\Omega = (0, 1)$. Here, suppose that $m > 0$ and $f : \Omega \to \mathbb{R}$ is continuous, and that u satisfies homogeneous Dirichlet boundary conditions. We want to simulate the heat equation for a time interval of length $T > 0$ using a finite element method.

(a) Describe the implicit Euler method. Identify the meaning of symbols used in the defining equation. State the implicit Euler method that makes clear what computational problem is to be solved at each time step.

(b) What is the discrete energy estimate satisfied by the implicit Euler method? What is the advantage over the explicit Euler method?

(c) Write down the Python code for the implicit Euler method for the computational problem with

$$f(x) = \sin(2\pi x)^2, \quad u_0(x) = 1.$$

This is a mock exam. You will be given a printed Python template, together with comments. You will have the same Python template on the computers, for testing.

1. Estimate the condition number of the linear system to be solved at each time step in terms of the eigenvalues of the matrices that you use.

Problem 4 (10 points)

Consider a one-dimensional advection-dominated problem:

$$-\partial_x^2 u + \beta \partial_x u + \gamma u = 1,$$
$$u(0) = 0, \quad u(1) = 0$$

over the interval $\Omega = (0, 1)$. Here, $\beta, \gamma \geq 0$.

(a) Describe the standard finite element method, using linear elements. What is the linear system of equations that you derive?

(b) Describe the numerical method that you obtain when adding artificial diffusion on each cell. Name all the variables used.

(c) Write down the Python code for finite element method with artificial diffusion for this problem. **This is a mock exam. You will be given a printed Python template, together with comments. You will have the same Python template on the computers, for testing.**