Numerical Approximation of PDEs

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Lecturer: Prof. Annalisa Buffa — Assistant: Mohamed Ben Abdelouahab

Project 1: The upwind quadrature scheme

We consider a stationary advection-diffusion equation modeling the temperature distribution u in a fluid streaming with velocity $\mathbf{b}$,

$$\begin{cases} -\epsilon\Delta u + \mathbf{b}\cdot\nabla u = 0 & \text{in } \Omega := (0,1)^2, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Assume that the velocity field is given by $\mathbf{b} = (-y, x)$, and the boundary datum is

$$g(x,y) = \begin{cases} \frac{1}{2} - \left|x - \frac{1}{2}\right|, & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Derive the weak formulation of problem (1) and prove that the problem admits a unique weak solution.

2. As $\epsilon \searrow 0$, the convection term will dominate the diffusion term (singular perturbation). What happens to the coercivity constant? Derive a piecewise linear finite element discretization of (1) and implement it in Python. Run a simulation for $\epsilon = 10^{-4}$. What do you observe?

3. Consider a numerical scheme where the diffusive part is discretized using standard piecewise linear finite elements, while the convective term is approximated using the so-called *upwind quadrature*, also known as Tabata scheme [1]. In this scheme, the local element contribution for the advection term is,

$$\int_T (\mathbf{b}\cdot\nabla u)v\,dxdy \approx \frac{|T|}{3}\sum_{\ell=1}^{3}(\mathbf{b}(p^\ell)\cdot(\nabla u)^+(p^\ell))v(p^\ell),$$

   where $\{p^\ell\}$ are the vertices of the triangle $T \in \mathcal{T}_h$, and where $(\nabla u)^+(p^\ell)$ is the gradient of u evaluated in p^ℓ from within the triangle lying upstream of T, namely in the direction of $-\mathbf{b}(p^\ell)$. Note that if $-\mathbf{b}(p^\ell)$ points along an edge of T, each neighboring triangle contribute a factor $1/2$.

   Let $\{\varphi_i\}_{i=1}^N$ be a basis of piecewise linear Lagrange finite elements on the triangulation $\mathcal{T}_h$. Show that the global convective term satisfies

$$\begin{aligned} b(\varphi_i,\varphi_j) &= \sum_{T\in\mathcal{T}_h:\, p^j\in T} \frac{|T|}{3}(\mathbf{b}(p^j)\cdot(\nabla\varphi_i)^+(p^j)), \qquad \forall\, i,j = 1,\dots,N \\ &= \begin{cases} m(p^j)(\mathbf{b}(p^j)\cdot\nabla\varphi_i|_{T'}(p^j)) & \text{if } T' \text{ is the upstream triangle,} \\ \frac{1}{2}m(p^j)(\mathbf{b}(p^j)\cdot\left(\nabla\varphi_i|_{T_1'} + \nabla\varphi_i|_{T_2'}\right)(p^j)) & \text{if } -\mathbf{b}(p^j) \text{ points along } T_1'\cap T_2' \end{cases} \end{aligned}$$

where mass of the j-th mesh node $m(p^j)$ is defined as

$$m(p^j) = \frac{1}{3} \sum_{T \in \mathcal{T}_h :\, p^j \in T} |T|. \tag{2}$$

4. Implement a Python function that computes the "local" contribution of a triangle T to the convective part of the system matrix. The function should take as input the element T defined by the 3×2 matrix of its vertices coordinates, a function handle to the velocity $\mathbf{b}$ and the vertex masses (2) which should be precomputed. For each vertex p^j:

   (a) Check if $-\mathbf{b}(p^j)$ is directed into the triangle T. You can transform $-\mathbf{b}(p^j)$ to the reference triangle.

   (b) If it does, compute $b(\varphi_{i|_T}, \varphi_{j|_T})$ for each i.

   (c) If it does not, ignore the current triangle, since it is not an upwind element.

   (d) If $-\mathbf{b}(p^j)$ points along an edge of T, divide by 2 the corresponding local contribution.

   Hint: The local contribution of a triangle T for a vertex $p^j \in T$, can be computed as $\alpha_T m(p^j)(\mathbf{b}(p^j) \cdot \nabla\varphi_i|_T(p^j))$ where $\alpha_T = 1$ if T is the upwind triangle, $\alpha_T = 0$ if it is not the upwind triangle, and $\alpha_T = \frac{1}{2}$ if $-\mathbf{b}(p^j)$ points along an edge of T.

5. Write a finite element code for the upwind quadrature approximation of (1) using the routine implemented in Task 4. How small $\epsilon > 0$ can the numerical scheme handle? Compare the numerical solutions obtained with the piecewise linear finite element discretization of Task 2 and the solution obtained with the upwind quadrature for $\epsilon = 10^{-4}$.

References

[1] Tabata M., *A finite element approximation corresponding to the upwind finite differencing*, Memoirs of Numerical Mathematics, No. 4, 1977.