Numerical Approximation of PDEs

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Project 2: River pollution problem in the advection dominating case

A factory spills a pollutant in the river Ω modeled as in Figure 1. The pollutant concentration C_{in} at the factory's inlet Γ_{in} is constant. The pollution does not involve the underwater layers of the river, so we consider this as a 2D problem in the xy-plane, with no dependence on the water depth. Moreover we assume that:

a. in correspondence of the upstream boundary Γ_{up} there is a small baseline concentration of pollutant C_{up};

b. the downstream section Γ_{down} is far enough to consider that the concentration does not change anymore in the direction of the flux normal to the boundary;

c. the flux of pollutant on the river sides is proportional to the difference between the natural concentration C_{dry} in the soil and the concentration in the river; we denote by α this constant of proportionality;

d. the diffusivity of the pollutant in the river is isotropic and constant, so it is represented by the scalar number μ;

e. the water velocity $\mathbf{u}$ in the river is steady and divergence-free;

f. a bacterium in the river reacts with the pollutant by destroying it with a rate σ;

g. the problem is steady, that is, no change in time.

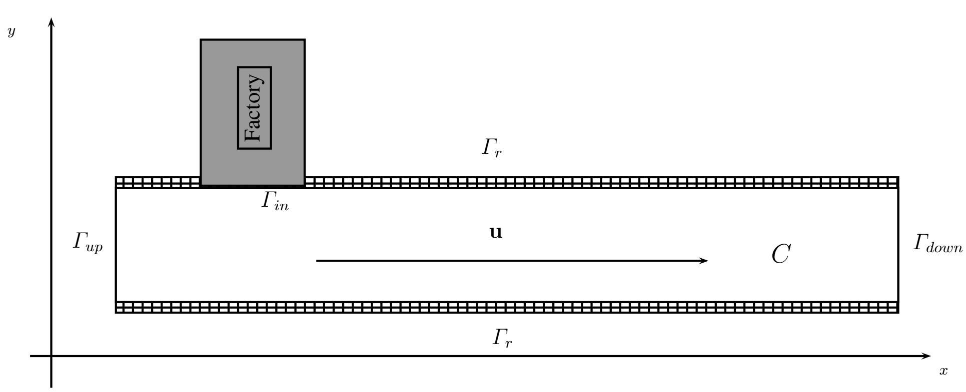

Figure 1: Computational domain Ω.

With this notation, being $C = C(x, y)$ the concentration of pollutant in the river, the differential problem for the steady dynamics of the pollutant is given by the following PDE:

$$\begin{cases} -\mu\Delta C + \mathbf{u}\cdot\nabla C + \sigma C = 0, & \text{in } \Omega \\ C = C_{in}, & \text{on } \Gamma_{in} \\ C = C_{up}, & \text{on } \Gamma_{up} \\ \mu\,\partial_n C = 0, & \text{on } \Gamma_{down} \\ \mu\,\partial_n C = \alpha(C_{dry} - C), & \text{on } \Gamma_r. \end{cases} \tag{1}$$

Assume that the river is rectilinear with length 10 meters and width 2 meters (hence in the x, y plane $\Omega = (0, 10) \times (0, 2)$), and that the location from which the factory spills pollutant is 1 meter down from the inlet and 2 meters wide. Data: river velocity $\mathbf{u} = [u_1, u_2]^T$ with $u_1 = u_M(2 - y)y$ m/s, $u_2 = 0$, $C_{up} = 1g/m^3$, $C_{in} = C_{up} + 15(1 - \cos(\pi(x - 1)))$g/m^3, $C_{dry} = C_{up}$, $\alpha = 0.1$.

Answer the following points

1. Analyze the well posedness of the PDE (1).
2. Discretize the problem with linear finite elements; choose a suitable grid (refined around the location of the factory) with size h, and simulate the following case:
   - $\sigma = 0.5$, $\mu = 10^{-6}$, $u_M = 10$.

   Consider the numerical solution C_h: what do you observe?
3. Consider the SUPG stabilisation as discussed in class with

   $$a_h(u, v) = a(u, v) + \sum_{K\in\mathcal{T}_h} \delta\frac{h_K}{\|\boldsymbol{\beta}\|}\int_K [\boldsymbol{\beta}\cdot\nabla u + \sigma u](\boldsymbol{\beta}\cdot\nabla u)$$

   and right-hand side

   $$R_h(v) = \int_\Omega fv + \sum_{K\in\mathcal{T}_h} \delta\frac{h_K}{\|\boldsymbol{\beta}\|}\int_K f\,(\boldsymbol{\beta}\cdot\nabla v) \quad \text{(+additional boundary integral terms)}$$

   Implement the SUPG stabilization in Python. Solve then the problem with the SUPG method and compare the results with the ones obtained with standard finite elements. Which differences do you notice?
4. Estimate numerically the convergence of $Q_h = \int_\Omega C_h$ to the exact value $Q = \int_\Omega C$. Since the exact solution is not known, compute a reference solution C_{ref} on a sufficiently refined grid. Is the observed convergence rate supported by the theory?

Detail all the answers and motivate all the choices you have made in the implementation.

References

[1] Hans-Görg Roos, Martin Stynes, Lutz Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer, 2008.

[2] A. Quarteroni, *Numerical Models for Differential Problems*, Springer, 2014.