Numerical Approximation of PDEs

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Project 3: The obstacle problem

Let $\Omega = (-1,1)^2$, $\varphi : \Omega \to \mathbb{R}$ be a regular and concave function, with $\varphi < 0$ on the border of Ω and $f \in L^2(\Omega)$. Next, define the energy functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$

Our goal is to find u, the minimizer of J in the set

$$\mathcal{U} = \{v \in H_0^1(\Omega) : v \geq \varphi \ a.e.\ \Omega\}.$$

If u denotes the vertical position of an elastic membrane, this problem allows us to compute the configuration that the membrane will assume when clamped on the boundary of Ω, subject to a downward force f and in part of the domain resting on an object shaped as φ. Related problems also arise in mathematical finance (e.g. pricing of American put options).

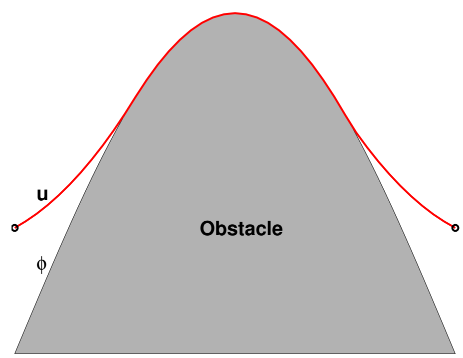

Figure 1: Schematic illustration in 1D

Answer the following points

1. Show that the minimization problem is equivalent to solving the variational inequality:

$$\text{Find } u \in \mathcal{U} \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u), \quad \forall v \in \mathcal{U}. \tag{1}$$

   Hint: see e.g. [2], Section 8.4.2, Theorem 4.

2. Show that in addition, problem (1) is equivalent to the following complementarity problem:

$$\begin{cases} -\Delta u \geq f & \text{in } \Omega, \\ u \geq \varphi & \text{in } \Omega, \\ (-\Delta u - f)(u-\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

   where the Laplacian operator and inequalities are taken in a weak sens.

   Hint: see e.g. [2], Section 8.4.2.

3. Let $\mathcal{T}_h$ be a mesh over Ω with N vertices, and consider a $\mathbb{P}1$ finite elements approximation of (2), resulting in the following system

$$\begin{cases} A\mathbf{u} \geq \mathbf{f}, \\ \mathbf{u} \geq \boldsymbol{\varphi}, \\ (A\mathbf{u} - \mathbf{f})^T(\mathbf{u} - \boldsymbol{\varphi}) = 0, \end{cases} \tag{3}$$

   where A is the usual matrix resulting from the discretization of the bilinear form $\int_\Omega \nabla u \nabla v$, and the inequalities hold element wise, e.g. $\mathbf{u}_i \geq \boldsymbol{\varphi}_i$ for every $i = 1, 2, \ldots, N$. Let us moreover denote D the diagonal of A. Problem (3) can be solved with an iterative method; here we consider the projected Jacobi method

$$\mathbf{u}^{k+\frac{1}{2}} = D^{-1}[\mathbf{f} - (A - D)\mathbf{u}^k], \qquad \mathbf{u}_i^{k+1} = \max\{\mathbf{u}_i^{k+\frac{1}{2}}, \boldsymbol{\varphi}_i\} \quad i = 1, \ldots, N,$$

and the related p-JOR ("projected Jacobi with over relaxation")

$$\mathbf{u}^{k+\frac{1}{2}} = \omega D^{-1}[\mathbf{f} - (A - D)\mathbf{u}^k] + (1-\omega)\mathbf{u}^k, \qquad \mathbf{u}_i^{k+1} = \max\{\mathbf{u}_i^{k+\frac{1}{2}}, \boldsymbol{\varphi}_i\} \quad i = 1, \ldots, N,$$

where ω is called "relaxation parameter" and can be chosen in the range $0 < \omega < \frac{2}{\rho(D^{-1}A)}$, $\rho(\cdot)$ being the spectral radius of a matrix (see e.g. [3]). When $\omega = 1$, one obtains the projected Jacobi method. Implement the p-JOR method. Verify numerically which is the maximum value of ω that ensure convergence of the method and which is the optimal value of ω. Assume $f = 0$ and $\varphi(x, y) = 0.5 - (x^2 + y^2)$.

4. Solve numerically the problem of the previous point with the p-JOR method using smaller and smaller mesh sizes. Derive numerically convergence rates for the error measured in the H^1 and L^2 norms. Analyse such convergence rates in view of the convergence theory proposed in [1, Section 5.1].

Suggested readings

[1] P. Ciarlet, "*The Finite Element Method for Elliptic Problems*", SIAM Classics in Applied Mathematics.

[2] L. C. Evans, "*Partial differential equations*", American Mathematical Society.

[3] A. Quarteroni, R. Sacco, F. Saleri, "*Numerical mathematics*", Springer.