Numerical Approximation of PDEs

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Project 4: Black-Scholes Equation: European Put

In mathematical finance, the Black-Scholes equation is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black-Scholes model. The Black-Scholes formula (also called Black-Scholes-Merton) was the first widely used model for option pricing. It's used to calculate the theoretical value of European-style options using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration and expected volatility[1].

The formula, developed by three economists – Fischer Black, Myron Scholes and Robert Merton – is perhaps the world's most well-known options pricing model. It was introduced in their 1973 paper, "The Pricing of Options and Corporate Liabilities," published in the Journal of Political Economy. Black passed away two years before Scholes and Merton were awarded the 1997 Nobel Prize in Economics for their work in finding a new method to determine the value of derivatives (the Nobel Prize is not given posthumously; however, the Nobel committee acknowledged Black's role in the Black-Scholes model).

Consider the Black & Scholes equation for the value $u(S,t)$ of an European Put option

$$\begin{cases} \partial_t u - \dfrac{\sigma^2}{2} S^2 \partial_{SS} u - rS\partial_S u + ru = 0 \quad \text{in } \Omega \times (0,T], \\ u(S,0) = u_0(S) = \max\{K - S, 0\}. \end{cases} \tag{1}$$

where $\Omega = (S_{\min}, S_{\max})$, σ and r are strictly positive and bounded constants, together with the following boundary conditions:

$$\partial_S u(S_{\min}, t) = u(S_{\max}, t) = 0.$$

We introduce the weighted Sobolev space V:

$$V = \left\{ v : v \in L^2(\Omega),\ S\frac{\partial v}{\partial S} \in L^2(\Omega), v(S_{\max}) = 0 \right\}.$$

Endowed with the inner product and norm

$$(v,w)_V = \int_\Omega \left(v(S)w(S) + S^2 \frac{\partial v}{\partial S}(S)\frac{\partial w}{\partial S}(S) \right) dS, \quad \|v\|_V = (v,v)_V^{1/2}.$$

The seminorm

$$|v|_V^2 = \int_\Omega \left(S\frac{\partial v}{\partial S} \right)^2 dS$$

is a norm in the Hilbert space V and we have the following Poincaré inequality:

$$\|v\|_{L^2(\Omega)} \leq 2|v|_V, \quad \forall v \in V. \tag{2}$$

Answer the following points:

a) Find the variational formulation of problem (1):

$$u \in \mathcal{C}^0\left((0,T]; L^2(\Omega)\right) \text{ satisfying}$$

$$\forall t \in (0,T), \quad \forall v \in V, \quad \left(\frac{\partial u}{\partial t}, v\right) + a(u,v) = 0 \quad \text{in } \Omega,$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega.$$

[1] In finance, an option is a contract which gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on a specified date, depending on the form of the option.

where $(\cdot,\cdot)$ denotes the L^2 inner product, and use the Poincaré inequality (2), to prove that there exists $\alpha > 0$ such that:

$$\forall t \in (0,T],\ \forall v \in V, \quad a(v,v) \geq \frac{\sigma^2}{4}|v|_V^2 - \alpha\|v\|_{L^2}^2.$$

b) Derive a semi-discretization formulation of (1). Using Lax-Milgram lemma, prove that the semi-discrete problem with the backward Euler is well-posed for a time step $\Delta t \leq \frac{1}{2\alpha}$, and we have the following estimate:

$$(1-2\alpha\Delta t)^N \|u^N\|_{L^2}^2 + \frac{\Delta t}{2}\sigma^2 \sum_{j=1}^{N} (1-2\alpha\Delta t)^{j-1}|u^j|_V^2 \leq \|u^0\|_{L^2}^2.$$

The sequence $\{u^j\}_j = \{u(S,t_j)\}_j$ corresponds to a uniform partition $\{t_j\}_{j=0}^N$ of the time interval $[0,T]$.

c) Implement both the backward Euler and the Crank–Nicolson schemes using $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements. Choose a smooth exact solution (by adding an appropriate right-hand side to equation (1) and adapting u_0) with the parameters $S_{\min} = 3$, $S_{\max} = 10$, $r = 0.04$, $\sigma = 0.2$, and $T = 1$, and perform a numerical convergence study.

Remark: For instance, the error for the $\mathbb{P}_1$ method with implicit Euler in time is expected to be $\mathcal{O}(h^2+\Delta t)$ in the L^2-norm. Here, we make sure that $\Delta t \approx h^2$ so that time discretization error does not dominate.

Now take $\Omega = (0,300)$ with parameters $K = 100$, $r = 0.04$, $\sigma = 0.2$, and $T = 5$. The exact solution of the Black–Scholes equation is given by the formula:

$$V(S,t) = N(-d_2)Ke^{-rt} - N(-d_1)S,$$

where $N(\cdot)$ is the normal CDF and

$$d_1(S,t) = \frac{1}{\sigma\sqrt{t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t\right], \quad d_2(S,t) = d_1 - \sigma\sqrt{t},$$

Are your results consistent with those obtained at the previous point? What would you suggest to improve the results?

Suggested Reading

[1] Achdou, Y. and Pironneau, O., 2005. *Computational methods for option pricing* (Vol. 30). Siam.

[2] Willmott, P., Howison, S. and Dewynne, J., 1995. "*The mathematics of financial derivatives: A student introduction*". Press Syndicate of the University of Cambridge., UK.