$N = \dim(V_h)$ $\qquad M = \dim(Q_h)$ $\qquad M < N$

$A \in \mathbb{R}^{N \times N}$ $\qquad B \in \mathbb{R}^{M \times N}$

$$\begin{pmatrix} \boxed{A} & \boxed{B^T} \\ \boxed{B} & 0 \end{pmatrix}$$

$\operatorname{Ker}(B) = \{ \underline{x} \in \mathbb{R}^N : B\underline{x} = 0 \}.$

$\operatorname{Im}(B) = \{ \underline{y} \in \mathbb{R}^M : \exists \underline{x} : B\underline{x} = \underline{y} \}.$

$\operatorname{Ker}(B^T) = \{ \underline{y} \in \mathbb{R}^M : B^T\underline{y} = 0 \}.$

$\operatorname{Im}(B^T) = \{ \underline{x} \in \mathbb{R}^N : \exists \underline{y} \in \mathbb{R}^M : B^T\underline{y} = \underline{x} \}.$

$$\boxed{\operatorname{Ker} B \underset{\mathbb{R}^N}{=} (\operatorname{Im} B^T)^\perp} \qquad \boxed{\operatorname{Ker}(B^T) \underset{\mathbb{R}^M}{=} (\operatorname{Im} B)^\perp}$$

$(\ )^\perp$ $\qquad Z \subseteq \mathbb{R}^N$ subspace

$$(Z)^\perp = \{ \underline{x} \in \mathbb{R}^N : \underline{x} \cdot \underline{z} = 0 \ \forall \underline{z} \in Z \}.$$

$Z \subseteq \mathbb{R}^N$ subspace.

$\Pi_Z : \mathbb{R}^N \to Z$ $\qquad$ projection from $\mathbb{R}^N$ to Z

$\Pi_Z^T : Z \to \mathbb{R}^N$ $\qquad$ identify elements of $\mathbb{R}^N$ into elements of Z.

<u>Conditions for uniqueness.</u>

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \qquad \begin{array}{ll} (1) & A\underline{x} + B^T\underline{y} = F \\ (2) & B\underline{x} = 0. \end{array}$$

$F = 0 \Rightarrow$ I want to find condition s.t. $\underline{x} = 0$, $\underline{y} = 0$.

Eq (2) $\quad B\underline{x} = 0 \quad \Rightarrow \quad \underline{x} \in \operatorname{Ker} B$

Eq (1) $\quad A\underline{x} + B^T\underline{y} = 0 \qquad K = \operatorname{Ker} B$

$$\Pi_K A\underline{x} + \Pi_K B^T y = 0$$

$\operatorname{Im} B^T = (\operatorname{Ker} B)^\perp \quad \Rightarrow \quad \Pi_K B^T y = 0.$

$$\Rightarrow \quad \boxed{\Pi_K A\underline{x} = 0 \ , \quad \underline{x} \in \operatorname{Ker} B} \qquad (*)$$

1. $(*)$ does not imply $\underline{x} = 0$.

We can show that uniqueness fails.

$\exists\, \underline{x}^* \in \operatorname{Ker} B \ : \ \Pi_K A\underline{x}^* = 0, \ \ \underline{x}^* \neq 0.$

$$A\underline{x}^* \in (\operatorname{Ker} B)^\perp = \operatorname{Im} B^T$$

$\exists\, y^* \in \mathbb{R}^M \ : \ A\underline{x}^* = B^T\underline{y}^*$

$$\begin{cases} A\underline{x}^* - B^T\underline{y}^* = 0. \\ B\underline{x}^* = 0 \end{cases} \qquad (x^*, -\underline{y}^*) \text{ is a non trivial solution.}$$

$\rightarrow$ uniq. fails.

2. (*) implies that $\underline{x}^* = 0$.

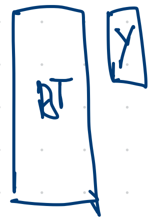
$$\{ \underline{x} \in K \ : \ \Pi_K A \underline{x} = 0 \} = \{0\}.$$

$$A\underline{x} + B^T \underline{y} = 0 \quad \Rightarrow \quad \underline{x} = 0. \quad B^T\underline{y} = 0$$

(if (*) is true)

In order to deduce $\underline{y} = 0$, we should have that $B^T \underline{y} = 0 \Rightarrow \underline{y} = 0$.

true $\Leftrightarrow$ B^T is full rank



If B^T is full rank, then the only solution of $Ax + B^T y = 0$, $Bx = 0$ is $\underline{x} = 0$, $\underline{y} = 0$.

$\rightarrow$ uniqueness is verified !

Conditions for uniqueness are then

1. $\underline{x} \in K \quad \Pi_K A \underline{x} = 0 \Rightarrow \underline{x} = 0$.
2. B^T - is full rank.
   is injective.

Let us elaborate on condition 1.

$$\Pi_K A \underline{x} = 0 \qquad \langle A\underline{x}, \underline{x}' \rangle = 0 \ \forall \underline{x}' \in K$$

$$\Updownarrow$$

$$\begin{pmatrix} A_{KK} & A_{KK^\perp} \\ A_{K^\perp K} & A_{K^\perp K^\perp} \end{pmatrix} \begin{pmatrix} \underline{x}_K \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \quad \underline{x}_K = 0.$$

$\Rightarrow$ A_{KK} must be invertible. (non-singular)

$\langle A\underline{x}, \underline{x}' \rangle$, $\underline{x}, \underline{x}' \in \ker B$ must be a coercive bilinear form.

Condition 1. is implied by:

$$\langle A\underline{x}, \underline{x} \rangle \geq \alpha_{(N,M)} \|\underline{x}\|_2^2 \qquad \underline{x} \in \ker B$$

$\hookrightarrow$ coercivity on the kernel.

1.

B^T is full rank if and only if

2.

$$\exists \beta > 0 \quad \forall \underline{y} \in \mathbb{R}^M : \quad \sup_{\underline{x} \in \mathbb{R}^N} \frac{\langle B^T \underline{y}, \underline{x} \rangle}{\|\underline{x}\|_2} > \beta \|\underline{y}\|_2$$

1. $\underline{x} \in \ker B$ corresponds to $\forall \underline{u}_h : b(\underline{u}_h, q_h) = 0 \ \forall q_h \in Q_h$

$\ker B \Leftrightarrow K_h = \{ \underline{u}_h \in V_h : b(\underline{u}_h, q_h) = 0 \ \forall q_h \in Q_h \}$.

$\langle A\underline{x}, \underline{x} \rangle = a(\underline{u}_h, \underline{u}_h) \qquad u_h \in K_h$.

$$a(\underline{u}_h, \underline{u}_h) \geq \alpha_h \|\underline{u}_h\|^2_{H^1(\Omega)} \qquad \mathbf{1'}$$

this condition, with α_h that depends on h, is equivalent to **1.**

2. $$\sup_{\underline{x} \in \mathbb{R}^N} \frac{\langle B^T \underline{y}, \underline{x} \rangle}{\|\underline{x}\|_2} > \beta \|\underline{y}\|_2$$

β depending on M, N

$$\underline{x} \in \mathbb{R}^N \iff \underline{u}_h \in X_h \qquad \underline{y} \in \mathbb{R}^M \iff q_h \in Q_h.$$

$$\langle B^T \underline{y}, \underline{x} \rangle = b(\underline{u}_h, q_h) = \int_\Omega \operatorname{div} \underline{u}_h \, q_h.$$

$$\sup_{\underline{u}_h \in V_h} \frac{b(\underline{u}_h, q_h)}{\|\underline{u}_h\|_{H^1(\Omega)}} > \beta_h \|q_h\|_{L^2(\Omega)} \qquad \mathbf{2'}$$

1. and **1'** are equivalent

2. and **2'** are equivalent !

1' and **2'** are the condition for uniqueness of the solution on the discrete problem !

- existence ✗ discrete ✗ infinite dim. level.
- stability ?

Stability is guaranteed when

$\exists \alpha > 0 \quad : \quad \alpha_h > \alpha > 0$

$\exists \beta > 0 \quad\quad \beta_h > \beta > 0$

1.' and 2.' are uniformely valid for all h, and with the choice of the norms already expressed in 1.' and 2.',

1.' COERCIVITY on the kernel

2.' INF-SUP condition / LBB condition.

Going back to Stokes

$$\begin{cases} -\Delta \underline{u} + \nabla p = f. \\ \quad \operatorname{div} \underline{u} = 0 \\ \quad \underline{u}|_{\partial\Omega} = 0. \end{cases}$$

Find $\underline{u} \in H_0^1(\Omega)^d$, $p \in L_0^2(\Omega)$

$$\int_\Omega \nabla \underline{u} \cdot \nabla \underline{v} + \int_\Omega p \operatorname{div} \underline{v} = \int f \underline{v}$$

$$\int_\Omega \operatorname{div} \underline{u}\, q = 0$$

$$\int \nabla p \cdot \underline{u} + \int \operatorname{div} \underline{u}\, p = 0$$

$p = \text{constant}$

$\Rightarrow \int \operatorname{div} \underline{u} = 0.$

$\hookrightarrow L_0^2(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q = 0 \}$

$$b(\underline{v}, p) = \int_\Omega \operatorname{div} \underline{u} \cdot p$$

$V_h \subseteq H^1_0(\Omega)^d \qquad Q_h \subseteq L^2_0(\Omega)$

About 1.' $\quad a(\underline{u}_h, \underline{v}_h) = \int_\Omega \nabla \underline{u}_h \cdot \nabla \underline{v}_h.$

is coercive over the whole space V_h !

$\Rightarrow$ it's coercive on all subspace of V_h.

$$a(\underline{u}_h, \underline{u}_h) \geq \alpha \, \| \underline{u}_h \|^2_{H^1(\Omega)^d}$$

About 2' $\quad \forall q_h \in Q_h \quad \sup_{\underline{u}_h \in V_h} \dfrac{\int_\Omega \operatorname{div} \underline{u}_h \cdot q_h}{\| \underline{u}_h \|_{H^1}} > \beta \, \| q_h \|_{L^2}$

- $\bar{q}_h$ * I look for a $\bar{\underline{u}}_h$ such that $\int_\Omega \operatorname{div} \bar{\underline{u}}_h \cdot \bar{q}_h > 0$
  - * I can prove some stability.

$$\| \bar{\underline{u}}_h \|_{H^1} \leq C \, \| \bar{q}_h \|_{L^2_0}$$

Let's try to prove 2.' on the simplest choice we can imagine

$$V_h = \{ \underline{u}_h \in H^1_0(\Omega) \quad \underline{u}_h|_T \in \mathbb{P}^1(T)^2 \quad \forall T \in \mathcal{T}_h \}$$

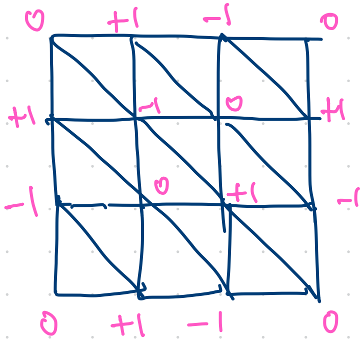
$$Q_h = \{ q_h \in L^2(\Omega) \quad \int_\Omega q_h = 0 \quad q_h|_T \in \mathbb{P}^1(T) \ \forall T \in \mathcal{T}_h \}$$

⚠ it does not work.

On a given mesh $\mathcal{T}_h$ $\quad \exists \, q_h$ such that I cannot find a specific $\underline{u}_h$ verifying $\int q_h \cdot \operatorname{div} \underline{u}_h > 0$

I'm going to show that

$$\exists\, q_h : \int q_h \operatorname{div} \underline{u}_h = 0 \quad \forall\, \underline{u}_h \in V_h$$



criss-cross mesh

q_h has the purple nodal values.

$$\int_\Omega q_h = 0 \quad ✓$$

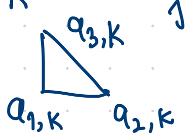
$\underline{u}_h \in V_h$ $\quad$ $\operatorname{div} \underline{u}_h \big|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h.$

$\mathbb{P}^1$

$$\left(\begin{matrix} a_0 + a_1 x + a_2 y \\ b_0 + b_1 x + b_2 y \end{matrix} \rightarrow a_1 + b_2 \right)$$

$$\int_\Omega q_h \cdot \operatorname{div} \underline{u}_h = \sum_T \int_T q_h \cdot \operatorname{div} \underline{u}_h = \sum_T (\operatorname{div} \underline{u}_h)\big|_T \int_K q_h$$

but $\displaystyle \int_K q_h = \sum_{j=0}^{2} q_h(a_{j,K}) = +1 - 1 + 0 \equiv 0.$!



$$\Rightarrow \int_\Omega q_h \operatorname{div} \underline{u}_h = \sum_T (\operatorname{div} \underline{u}_h)\big|_T \cdot 0 \equiv 0$$

true for all $\underline{u}_h$.

B^T has a kernel, it can't be full rank.

q_h that we just constructed is a spurious mode in the system :-(

Another choice of spaces.

V_h as before

$Q_h^0 = \{ q_h \in L^2(\Omega) \quad : \quad q_h|_T \in \mathbb{P}^0(T) \}$.

1.' is still true

2.' remains to be studied..