Numerical Approximation of PDEs

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Exercise 1. [The MINI Element for the steady Stokes problem]

- Consider the Stokes problem:

$$\begin{cases} -\Delta \underline{u} + \nabla p &= \underline{f} \quad \text{in } \Omega, \\ \nabla \cdot \underline{u} &= 0 \quad \text{in } \Omega, \\ \underline{u} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

  Write the variational formulation.

- If you replace the Hilbert spaces with finite dimensional spaces V_h for $\underline{u}$ and Q_h for p; then the discrete problem takes the form of the following linear system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

  To what corresponds the matrices A and B ?

- Let $\mathcal{T}_h$ be a conforming triangulation of $\Omega \subset \mathbb{R}^2$. For each triangle $T \in \mathcal{T}_h$, define the bubble function:

$$b_T(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x),$$

  where $\lambda_i(x)$ are the barycentric coordinates on T.

  Define the velocity and pressure spaces as:

$$V_h = \left\{ \underline{u}_h \in [H_0^1(\Omega)]^2 : \underline{u}_h|_T \in [\mathbb{P}^1(T) \oplus \text{span}(b_T)]^2, \forall T \in \mathcal{T}_h \right\},$$
$$Q_h = \left\{ q_h \in L_0^2(\Omega) \cap C^0(\Omega) : q_h|_T \in \mathbb{P}^1(T), \forall T \in \mathcal{T}_h \right\}.$$

  Prove that with this choice of V_h and Q_h (known as the MINI Element), the matrix $B \in \mathbb{R}^{M \times N}$ has full rank.

  Note that: $\dim(V_h) = \#V_0 + \#T = N > \dim(Q_h) = \#V - 1 = M$, where

  - $\#V$ is the total number of vertices in $\mathcal{T}_h$.
  - $\#V_0$ is the number of interior vertices.
  - $\#T$ is the number of triangles in $\mathcal{T}_h$.

NB. For further details, see the book by Boffi, Brezzi and Fortin.

Solution:

- *We seek* $\underline{u} \in V := [H_0^1(\Omega)]^2$ *and* $p \in Q := L_0^2(\Omega)$ *such that:*

$$\int_\Omega \nabla \underline{u} : \nabla \underline{v} \, dx - \int_\Omega p \, \nabla \cdot \underline{v} \, dx = \int_\Omega \underline{f} \cdot \underline{v} \, dx \quad \forall \underline{v} \in V,$$
$$\int_\Omega q \, \nabla \cdot \underline{u} \, dx = 0 \quad \forall q \in Q.$$

- *Let* $V_h \subset V$ *and* $Q_h \subset Q$ *be finite-dimensional subspaces. The discrete problem takes the form:*

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

  where:

  - $A_{ij} = \int_\Omega \nabla \phi_j : \nabla \phi_i \, dx$ *corresponds to the stiffness matrix for the velocity space* V_h*,*
  - $B_{ij} = -\int_\Omega \psi_i \, \nabla \cdot \phi_j \, dx$ *represents the discretization of the divergence operator,*
  - $F_i = \int_\Omega \underline{f} \cdot \phi_i \, dx$ *is the load vector.*

  Here, $\{\phi_j\}$ *is a basis for* V_h *and* $\{\psi_i\}$ *is a basis for* Q_h*.*

- *We want to prove that the matrix* B *has full rank. This is needed to show that the discrete inf-sup (Ladyzhenskaya–Babuška–Brezzi) condition holds for the pair* (V_h, Q_h)*. That is, there exists a constant* $\beta > 0$*, independent of* h*, such that:*

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\underline{v}_h \in V_h \setminus \{0\}} \frac{\int_\Omega q_h \nabla \cdot \underline{v}_h \, dx}{\|\underline{v}_h\|_{H^1} \|q_h\|_{L^2}} \geq \beta.$$

  A sufficient condition for B *to have full rank is to show that* $\forall q_h \in Q_h$*,* $\exists \underline{v}_h \in V_h$ *such that:*

$$\int_\Omega q_h \nabla \cdot \underline{v}_h \, dx \neq 0.$$

  Integration by parts gives:

$$-\int_\Omega q_h \nabla \cdot \underline{v}_h \, dx = \int_\Omega \underline{v}_h \cdot \nabla q_h \, dx - \int_{\partial\Omega} \underbrace{v_h \cdot n}_{=0} \, q_h.$$

  Since q_h *is continuous and piecewise linear on* $\mathcal{T}_h$*, we have* $\nabla q_h|_T \in [\mathbb{P}_0(T)]^2$*.*

  Now, let us construct a test function $\underline{v}_h \in V_h$ *supported in each triangle* T *using the bubble function* $b_T(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)$*.*

  Define, for a fixed triangle T*, the local function:*

$$\underline{v}_h^T = b_T \, \nabla q_h|_T.$$

  Note that $\underline{v}_h^T \in [b^T]^2$ *and vanishes on* ∂T*, so* $\underline{v}_h^T \in V_h$*.*

  Then:

$$\int_T \underline{v}_h^T \cdot \nabla q_h \, dx = |\nabla q_h|^2 \int_T b_T \, dx \geq 0 \qquad \text{since } \nabla q_h \text{ is constant on } T.$$

To construct a global function, define:

$$\underline{v}_h(x) = \sum_{T \in \mathcal{T}_h} \underline{v}_h^T(x).$$

This function is in V_h because each $\underline{v}_h^T$ is supported on T and belongs to $[P^1(T) \oplus \mathrm{span}(b_T)]^2$.

Now :

$$\int_\Omega \underline{v}_h \cdot \nabla q_h \, dx = \sum_{T \in \mathcal{T}_h} \int_T \underline{v}_h^T \cdot \nabla q_h \, dx = \sum_{T \in \mathcal{T}_h} |\nabla q_h|^2 \int_T b_T(x) \, dx.$$

Since each term in the sum is non-negative and at least one term is strictly positive whenever $q_h \neq 0$, we conclude:

$$\int_\Omega \underline{v}_h \cdot \nabla q_h \, dx \neq 0.$$

This proves that B has full rank for the MINI Element.