Numerical Approximation of PDEs

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Exercise 1. We consider the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \text{ in } \Omega \times (0, T], \\ u = 0 \text{ on } \partial\Omega \times (0, T], \\ u(0) = u_0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial t}(0) = v_0 \text{ in } \Omega. \end{cases}$$

Given a uniform grid of $[0, T]$

$$0 = t^0 < t^1 < ... < t^N = T,$$

the fully discrete approximation of the wave equation with Finite Elements of degree r and the explicit Newmark method to advance in time reads : starting from $u_h^0 = I_h^r u_0$ and $v_h^0 = I_h^r v_0$ solve for $n = 0, 1, 2, ..., N$

$$\int_\Omega \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} v_h + c^2 \int_\Omega \nabla u_h^n \nabla v_h = 0, \forall v_h \in X_{h,0}^r, n \geq 1,$$

with the initialization

$$\int_\Omega \frac{u_h^1 - u_h^0}{\Delta t} v_h + \frac{c^2 \Delta t}{2} \int_\Omega \nabla u_h^0 \nabla v_h = \int_\Omega v_h^0 v_h, \forall v_h \in X_{h,0}^r.$$

Prove that the Newmark method is stable under the CFL condition

$$c\Delta t \leq \frac{1}{C_I} h$$

where C_I is the constant of the inverse inequality. In particular there exists a pure real constant $C > 0$ such that for all $m = 1, 2, ..., N$

$$\left\| \frac{u_h^m - u_h^{m-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^m\|_{L^2(\Omega)}^2 \leq C \left(\|v_h^0\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^0\|_{L^2(\Omega)}^2 \right).$$

Hint 1: For the first step, choose $v_h = \frac{u_h^1 - u_h^0}{\Delta t}$ and prove that under the CFL condition it holds

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^1\|_{L^2(\Omega)}^2 \le C \left(\|v_h^0\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^0\|_{L^2(\Omega)}^2 \right)$$

where $C > 0$ is a pure real constant.

Hint 2: For $n \ge 1$, choose $v_h = \frac{u_h^{n+1} - u_h^n}{\Delta t} + \frac{u_h^n - u_h^{n-1}}{\Delta t}$ as for the proof in the implicit case, and observe that

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} + \frac{u_h^n - u_h^{n-1}}{\Delta t} = \frac{u_h^{n+1} - u_h^{n-1}}{\Delta t}.$$

Then note that

$$\int_\Omega \nabla u_h^n \nabla (u_h^{n+1} - u_h^{n-1}) = \int_\Omega \nabla u_h^{n+1} \nabla u_h^n - \int_\Omega \nabla u_h^n \nabla u_h^{n-1}$$

is a telescopic sum.

Solution:

Case 1 $n = 0$: If we choose $v_h = \frac{u_h^1 - u_h^0}{\Delta t}$, we have

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \int_\Omega \nabla u_h^0 \nabla (u_h^1 - u_h^0) = \int_\Omega v_h^0 \frac{u_h^1 - u_h^0}{\Delta t}.$$

Using the identity $ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$, we obtain

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{c^2}{4} \|\nabla u_h^1\|_{L^2(\Omega)}^2 - \frac{c^2}{4} \|\nabla u_h^0\|_{L^2(\Omega)}^2 = \int_\Omega v_h^0 \frac{u_h^1 - u_h^0}{\Delta t} + \frac{\Delta t^2 c^2}{4} \left\| \frac{\nabla u_h^1 - \nabla u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2.$$

By Cauchy-Schwarz and inverse inequality it yields

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{c^2}{4} \|\nabla u_h^1\|_{L^2(\Omega)}^2 - \frac{c^2}{4} \|\nabla u_h^0\|_{L^2(\Omega)}^2 \le \|v_h^0\|_{L^2(\Omega)} \left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)} + \frac{\Delta t^2 c^2 C_I^2}{4h^2} \left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2.$$

By Young's inequality we have

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{c^2}{4} \|\nabla u_h^1\|_{L^2(\Omega)}^2 - \frac{c^2}{4} \|\nabla u_h^0\|_{L^2(\Omega)}^2 \le \|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{\Delta t^2 c^2 C_I^2}{4h^2} \left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2.$$

By using the CFL condition, we can pass the last term of the RHS to the LHS and we obtain

$$\frac{1}{4} \left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{c^2}{4} \|\nabla u_h^1\|_{L^2(\Omega)}^2 - \frac{c^2}{4} \|\nabla u_h^0\|_{L^2(\Omega)}^2 \le \|v_h^0\|_{L^2(\Omega)}^2.$$

Thus finally, multiplying by 4 we obtain

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^1\|_{L^2(\Omega)}^2 \le 4 \|v_h^0\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^0\|_{L^2(\Omega)}^2.$$

Case 2 : $n \geq 1$ Observing that

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} = \frac{1}{\Delta t}\left(\frac{u_h^{n+1} - u_h^n}{\Delta t} - \frac{u_h^n - u_h^{n-1}}{\Delta t}\right).$$

Thus choosing $v_h = \frac{u_h^{n+1} - u_h^n}{\Delta t} + \frac{u_h^n - u_h^{n-1}}{\Delta t}$ we have

$$\frac{1}{\Delta t}\left\|\frac{u_h^{n+1} - u_h^n}{\Delta t}\right\|_{L^2(\Omega)}^2 - \frac{1}{\Delta t}\left\|\frac{u_h^n - u_h^{n-1}}{\Delta t}\right\|_{L^2(\Omega)}^2 + c^2\int_\Omega \nabla u_h^n \left(\frac{\nabla u_h^{n+1} - \nabla u_h^{n-1}}{\Delta t}\right) = 0.$$

Multiplying by Δt and rewritting the gradient term as a telescopic sum, we have

$$\left\|\frac{u_h^{n+1} - u_h^n}{\Delta t}\right\|_{L^2(\Omega)}^2 - \left\|\frac{u_h^n - u_h^{n-1}}{\Delta t}\right\|_{L^2(\Omega)}^2 + c^2\int_\Omega \nabla u_h^{n+1}\nabla u_h^n - c^2\int_\Omega \nabla u_h^n \nabla u_h^{n-1} = 0.$$

Summing up over $n = 1, ..., m-1$ we obtain

$$\left\|\frac{u_h^m - u_h^{m-1}}{\Delta t}\right\|_{L^2(\Omega)}^2 - \left\|\frac{u_h^1 - u_h^0}{\Delta t}\right\|_{L^2(\Omega)}^2 + c^2\int_\Omega \nabla u_h^m \nabla u_h^{m-1} - c^2\int_\Omega \nabla u_h^1 \nabla u_h^0 = 0.$$

Thus

$$\left\|\frac{u_h^m - u_h^{m-1}}{\Delta t}\right\|_{L^2(\Omega)}^2 + c^2\int_\Omega \nabla u_h^m \nabla u_h^{m-1} = \left\|\frac{u_h^1 - u_h^0}{\Delta t}\right\|_{L^2(\Omega)}^2 + c^2\int_\Omega \nabla u_h^1 \nabla u_h^0.$$

By using again the identity $ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$ and Young's inequality yields

$$\begin{aligned}
&\left\|\frac{u_h^m - u_h^{m-1}}{\Delta t}\right\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u_h^m\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u_h^{m-1}\|_{L^2(\Omega)}^2 \\
&\quad \leq \left\|\frac{u_h^1 - u_h^0}{\Delta t}\right\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u_h^1\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u_h^0\|_{L^2(\Omega)}^2 + \frac{c^2\Delta t^2}{2}\left\|\frac{\nabla u_h^m - \nabla u_h^{m-1}}{\Delta t}\right\|_{L^2(\Omega)}^2
\end{aligned}$$

We conclude as in the first step by using the inverse inequality and the CFL condition, and we use the case 1 to bound

$$\left\|\frac{u_h^1 - u_h^0}{\Delta t}\right\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\|\nabla u_h^1\|_{L^2(\Omega)}^2.$$