Short notes on functional analysis

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1 Banach and Hilbert spaces

1.1 Norms

Let V be a linear space (i.e. a vector space) over $\mathbb{R}$. A **seminorm** on V is a map $\|\cdot\| : V \to \mathbb{R}$ such that

1. $\|v\| \geq 0 \quad \forall v \in V$;
2. $\|\alpha v\| = |\alpha|\|v\| \quad \forall \alpha \in \mathbb{R}, \quad \forall v \in V$;
3. $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$ (triangular inequality).

A **norm** on V is a seminorm satisfying the condition that

$$\|v\| = 0 \quad \text{if and only if} \quad v = 0.$$

The pair $(V, \|\cdot\|)$ is a **normed space**, and we can define a distance function d on V via $d(u, v) = \|u - v\|$.

Example 1. For the space $\mathbb{R}^d$, we can define the *Euclidean norm*

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}} \qquad \forall \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Two norms $\|\cdot\|$ and $|||\cdot|||$ on a linear space V are **equivalent**, if there exist two positive constants C_1 and C_2 such that

$$C_1\|x\| \leq |||x||| \leq C_2\|x\| \qquad \forall x \in V.$$

1.2 Sequences and Banach spaces

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in a normed space $(V, \|\cdot\|)$. The sequence is said to be

- a **Cauchy** sequence, if
$$\lim_{n,m\to\infty} \|u_n - u_m\| = 0 \ ,$$
- a **convergent** sequence, if
$$\lim_{n\to\infty} \|u_n - u\| = 0 \text{ with } u \in V \ .$$

A normed space $(V, \|\cdot\|)$ is called a **Banach space**, if any Cauchy sequence in V converges to an element of V (with respect to the $\|\cdot\|$ norm). In other words, a Banach space is a complete normed vector space.

1.3 Scalar product and Hilbert spaces

A **scalar product** in on linear space V is a form $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ such that:

1. $(u,u) \geq 0 \quad \forall u \in V$ and $(u,u) = 0 \Leftrightarrow u = 0$;
2. $(u,v) = (v,u) \quad \forall u,v \in V$;
3. $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w) \quad \forall u,v,w \in V, \quad \alpha,\beta \in \mathbb{R}$.

We denote by $(V,(\cdot,\cdot))$ a linear space with scalar product. Two vectors $v, w \in V$ are said to be **orthogonal** if $(v,w) = 0$. Moreover, we have the Cauchy–Schwarz inequality.

Theorem 1.1 (Cauchy–Schwarz inequality)**.** *Let $(V,(\cdot,\cdot))$ be a linear space with scalar product. Then,*

$$|(u,v)| \leq \sqrt{(u,u)}\sqrt{(v,v)} \qquad \forall u,v \in V.$$

Observe that a scalar product induces a norm in natural way. Furthermore, as a consequence of the Cauchy–Schwarz inequality we get the following result.

Theorem 1.2. *Let $(V,(\cdot,\cdot))$ be a linear space with scalar product, and set $\|v\| = \sqrt{(v,v)}$ for all $v \in V$. Then, the pair $(V,\|\cdot\|)$ is a normed space and*

$$|(u,v)| \leq \|u\|\|v\| \qquad \forall u,v \in V.$$

A linear space with scalar product is said to be **pre-Hilbertian**. A pre-Hilbertian space that is Banach with respect to a norm induced by a scalar product is called a **Hilbert space**. Tools like orthonormal bases and projections are well defined in Hilbert spaces.

Example 2. The space $(\mathbb{R}^d, \|\cdot\|_2)$ is a Hilbert space with norm

$$\|\mathbf{x}\|_2 = \sqrt{(\mathbf{x},\mathbf{x})} = \sqrt{\sum_{i=1}^{d} x_i^2} \qquad \forall \mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d,$$

where $(\cdot,\cdot)$ is the *Euclidean* scalar product.

2 The L^p spaces

Let the **space** $L^p(\Omega)$, $1 \leq p < \infty$, be defined as follows:

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ Lebesgue measurable } : \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

More precisely, L^p are spaces of *equivalence classes* of measurable functions: f and g belong to the same equivalence class, if they differ at most on a set of zero measure ("they are equal almost everywhere"). In other words, if f and g are two $L^p(\Omega)$ functions whose difference is non-zero only on a zero-measure set, then f and g are "indistinguishable" in the L^p topology and should be identified as just one function (pretty much as $2/3, 4/6, 6/9$ represent the same rational number). Therefore, it does not make sense to look at the value of an L^p function on a zero-measure set. Consequently, it does not make sense to talk about point values for elements of an L^p space.

For any $1 \le p < \infty$, the L^p space is a Banach space when equipped with the integral norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} .$$

Furthermore, L^2 is a Hilbert space endowed with the scalar product

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx,$$

which induces the norm

$$\|f\|_{L^2(\Omega)} = \sqrt{\int_\Omega |f(x)|^2 \, dx}.$$

Finally, we denote by $L^\infty(\Omega)$ the set of equivalence classes of essentially bounded functions, i.e. functions that are unbounded at most on a set of measure zero. The space L^∞ is also a Banach space when equipped with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} f(x) = \inf\{C \ge 0 : |f(x)| \le C \text{ for almost every } x \in \Omega\}.$$

Theorem 2.1. *If Ω is a bounded domain, then $L^\infty(\Omega) \subset \ldots \subset L^p(\Omega) \ldots \subset L^2(\Omega) \subset L^1(\Omega)$.*

3 Distributions

Let $\mathcal{D}(\Omega)$ denote the space of functions with compact support (i.e. they are identically zero outside some compact subset $K \subset \Omega$) that admit infinite derivatives, i.e. $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$. A sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ **converges** to $\phi \in \mathcal{D}(\Omega)$ **in $\mathcal{D}(\Omega)$** if:

1. there exists a compact subset $K \subset \Omega$ such that

$$\operatorname{supp} \phi_n \subset K \qquad \forall n \in \mathbb{N} ,$$

2. there holds

$$\partial^{\boldsymbol{\alpha}} \phi_n \longrightarrow \partial^{\boldsymbol{\alpha}} \phi \qquad \forall \boldsymbol{\alpha} \in \mathbb{N}^d ,$$

with respect to the supremum norm $\|\cdot\|_\infty$. Here, we have used the multi-index notation $\partial^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$ for any multi-index $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$. We note that $\partial^{\boldsymbol{\alpha}}$ is sometimes also denoted by $D^{\boldsymbol{\alpha}}$.

Let T be a linear operator (i.e. a map) from $\mathcal{D}(\Omega)$ to $\mathbb{R}$. We denote by $\langle T, \varphi \rangle$ the value obtained by applying T to $\varphi \in \mathcal{D}(\Omega)$, i.e. $\langle T, \varphi \rangle = T(\varphi)$. We say that T is **continuous**, if

$$\lim_{n \to \infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle$$

for any sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ that converges to φ in $\mathcal{D}(\Omega)$. A continuous linear operator $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a **distribution**. The **space of distributions** on Ω is denoted by $\mathcal{D}'(\Omega)$.

A **sequence of distributions** $\{T_n\}$ **converges** in $\mathcal{D}'(\Omega)$ to a distribution T if

$$\lim_{n \to \infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega) .$$

Example 3. For any function $f \in L^1_{\text{loc}}(\Omega)$ it is natural to associate a distribution $T_f \in \mathcal{D}'(\Omega)$ via

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,dx\ , \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{1}$$

Example 4. Let $\omega \in \Omega$. The **Dirac delta** δ_ω associated with the point ω is the distribution defined by the point evaluation

$$\langle \delta_\omega, \varphi \rangle = \varphi(\omega)\ , \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

To present the intuition behind this definition, we first introduce the **characteristic function** on an interval $[a, b]$, which is given by

$$\chi_{[a,b]}(x) = \begin{cases} 1\ , & \text{if } x \in [a,b]\ , \\ 0\ , & \text{elsewhere}\ . \end{cases}$$

It can be easily seen that the sequence of distributions associated with $f_n = \frac{n}{2}\chi_{[\omega-1/n,\omega+1/n]}$ converges to the distribution δ_ω, so that we can recover the "intuitive" definition of δ_ω (that is, δ_ω viewed as a "function", which is zero everywhere except in $x = \omega$ where its value is infinite and whose integral is one nonetheless). Indeed, in view of the previous example we find that

$$\langle T_{f_n}, \varphi \rangle = \frac{n}{2}\int_{\omega-\frac{1}{n}}^{\omega+\frac{1}{n}} \varphi(x)\,dx = \frac{n}{2}\left(\Phi(\omega + 1/n) - \Phi(\omega - 1/n)\right) \qquad \forall \varphi \in \mathcal{D}(\Omega)\ ,$$

where Φ denotes the primitive of φ. Setting $h = 1/n$, we have

$$\langle T_{f_n}, \varphi \rangle = \frac{\Phi(\omega + h) - \Phi(\omega - h)}{2h}\ ,$$

which converges to $\Phi'(\omega) = \varphi(\omega)$ as $h \to 0$ (or, equivalently, $n \to \infty$). Thus,

$$\lim_{n\to\infty} \langle T_{f_n}, \varphi \rangle = \varphi(\omega) = \langle \delta_\omega, \varphi \rangle\ ,$$

as claimed. That is, $T_{f_n} \longrightarrow \delta_\omega$ in $\mathcal{D}'(\Omega)$. We remark that δ_ω is not a function in a classical sense, i.e. it is not possible to find a function $f \in L^1_{\text{loc}}(\Omega)$ such that the associated distribution T_f satisfies

$$\langle T_f, \varphi \rangle = \varphi(\omega)\ , \qquad \forall \varphi \in \mathcal{D}(\Omega)\ .$$

3.1 Derivatives of distributions

To define the derivative of a distribution, it is desirable to choose a definition that provides the property that $(T_f)' = T_{f'}$, provided f is sufficiently smooth. For example, if $\Omega \subset \mathbb{R}$ and $\varphi \in \mathcal{D}(\Omega)$, then integration by parts yields

$$\langle T_{f'}, \varphi \rangle = \int_\Omega f'(x)\varphi(x)\,dx = -\int_\Omega f(x)\varphi'(x)\,dx\ ,$$

provided f is sufficiently smooth. This formal calculation eventually motivates the following definition.

Let Ω be an open subset of $\mathbb{R}^d$ and let $T \in \mathcal{D}'(\Omega)$. The **partial derivative** of T with respect to x_i, $1 \le i \le d$, **in the sense of distributions** is the distribution defined through

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega)\ .$$

Similarly, we can define higher-order derivatives. In fact, for any multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, the derivative of T is defined through

$$\langle \partial^{\boldsymbol{\alpha}} T, \varphi \rangle = (-1)^{|\boldsymbol{\alpha}|} \langle T, \partial^{\boldsymbol{\alpha}} \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega) \ .$$

Example 5 (Important!)**.** The **Heaviside function** Heav on $\mathbb{R}$ is defined by

$$\mathrm{Heav}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Its derivative in the sense of distributions is the Dirac distribution associated to the point $x = 0$, i.e.

$$\mathrm{Heav}' = \delta_0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).$$

Indeed, applying the definition of the distributional derivative, we find that

$$\langle \mathrm{Heav}', \varphi \rangle = -\langle \mathrm{Heav}, \varphi' \rangle = -\int_{-\infty}^{\infty} \mathrm{Heav}(x) \varphi'(x) \, dx = -\int_0^{\infty} \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle$$

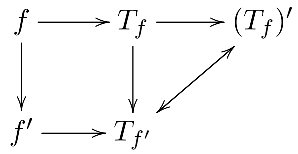
for every $\varphi \in \mathcal{D}(\mathbb{R})$. That is, $\mathrm{Heav}' = \delta$, implying that Heav' cannot be a function!

Example 6. Let $\Omega \subset \mathbb{R}$ and $[a, b] \subset \Omega$, and let $\chi_{[a,b]}$ be the characteristic function on $[a, b]$. Its derivative in the sense of distributions is $\delta_a - \delta_b$. Indeed, for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle T'_{\chi_{[a,b]}}, \varphi \rangle = -\langle T_{\chi_{[a,b]}}, \varphi' \rangle = -\int_\Omega \chi_{[a,b]} \varphi'(x) \, dx = -\int_a^b \varphi'(x) \, dx = -\varphi(b) + \varphi(a) = \langle \delta_a, \varphi \rangle - \langle \delta_b, \varphi \rangle \ .$$

The set $\mathcal{D}'(\Omega)$ is closed with respect to taking derivatives (in the sense of distributions). This means that **every distribution has infinite derivatives**. It can be shown that the derivative operator (in the sense of distributions) is continuous: indeed, if $T_n \longrightarrow T$ in $\mathcal{D}'(\Omega)$, then $\partial^{\boldsymbol{\alpha}} T_n \longrightarrow \partial^{\boldsymbol{\alpha}} T$ in $\mathcal{D}'(\Omega)$ for all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^d$.

As motivated at the beginning of this subsection, the derivative in the sense of distributions is indeed a generalisation of the classical derivative of functions. In fact, if a function f is of class C^1, then the derivative of its associated distribution T_f coincides with the distribution $T_{f'}$ associated to the classical derivative f' of f. We can schematically summarise this idea as follows:



4 Sobolev spaces

It is immediate to see that if Ω is a bounded domain, then L^2 functions are distributions, $L^2(\Omega) \subset \mathcal{D}'(\Omega)$. However, it is not guaranteed that their derivatives (in the sense of distributions) will still belong to $L^2(\Omega)$ (or even that they will be functions, see ,e.g., Heav').

These considerations motivate to introduce **Sobolev spaces** $W^{k,p}$ of order $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ by

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega) : \partial^{\boldsymbol{\alpha}} u \in L^p(\Omega) \ \forall |\boldsymbol{\alpha}| \leq k \} \, ,$$

where the derivatives $\partial^{\boldsymbol{\alpha}} f$ are to be interpreted **in the sense of distributions**. The spaces $W^{k,p}$ are Banach spaces when equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\boldsymbol{\alpha}|\le k} \|\partial^{\boldsymbol{\alpha}} u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \le p < +\infty \ , \\ \max_{|\boldsymbol{\alpha}|\le k} \|\partial^{\boldsymbol{\alpha}} u\|_{L^\infty(\Omega)}\,, & p = +\infty. \end{cases}$$

Of special interest are the Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$, which are thus given by

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \,:\, \partial^{\boldsymbol{\alpha}} f \in L^2(\Omega), \quad |\boldsymbol{\alpha}| \le k \right\} \ .$$

In particular, $H^{k+1}(\Omega) \subset H^k(\Omega)$ for $k \ge 0$, with $H^0(\Omega) = L^2(\Omega)$. The space $H^k(\Omega)$ is a Hilbert space with respect to the scalar product

$$(f, g)_{H^k(\Omega)} = \sum_{|\boldsymbol{\alpha}|\le k} \int_\Omega \partial^{\boldsymbol{\alpha}} f(x) \partial^{\boldsymbol{\alpha}} g(x)\, dx \ ,$$

which induces the norm

$$\|f\|_{H^k(\Omega)} = \sqrt{(f,f)_{H^k(\Omega)}} = \sqrt{\sum_{|\boldsymbol{\alpha}|\le k} \int_\Omega |\partial^{\boldsymbol{\alpha}} f(x)|^2\, dx} \ .$$

We also define the seminorm $|\cdot|_{H^k(\Omega)}$ in $H^k(\Omega)$ as follows:

$$|f|_{H^k(\Omega)} = \sqrt{\sum_{|\boldsymbol{\alpha}|= k} \int_\Omega |\partial^{\boldsymbol{\alpha}} f(x)|^2\, dx} \ .$$

Then we have in particular that

$$\|f\|_{H^k(\Omega)} = \sqrt{\sum_{m=0}^{k} |f|^2_{H^m(\Omega)}}.$$

4.1 Regularity of functions in $H^k(\Omega)$

The following example shows that a function in $H^1(\Omega)$ is, in general, not continuous if Ω is an open set in $\mathbb{R}^2$.

Example 7. Let $\Omega \subset \mathbb{R}^2$ be the disk of radius $r < 1$ centred in the origin. Then the function f defined in $\Omega \backslash \{(0,0)\}$ as

$$f(x,y) = \left| \ln \frac{1}{\sqrt{x^2+y^2}} \right|^\alpha \ ,$$

with $0 < \alpha < 1/2$, belongs to $H^1(\Omega)$, but it is not continuous in $(0,0)$.

Concerning the regularity of the functions in $H^k(\Omega)$, the following general result holds.

Theorem 4.1 (Sobolev embedding theorem)**.** *Let Ω be an open subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary. Then, the following continuous embeddings hold:*

1. *if* $0 \le 2k < d$, *then* $H^k(\Omega) \subset L^{p^*}(\Omega)$, $p^* = 2d/(d-2k)$;

2. *if* $2k = d$, *then* $H^k(\Omega) \subset L^q(\Omega)$, $2 \le q < \infty$;

3. *if* $2(k-m) > d$, *then* $H^k(\Omega) \subset C^m(\overline{\Omega})$.

Remark 4.1. *In particular, if* $d = 1$*, then functions in* $H^1(\Omega)$ *are continuous. If* $d \in \{2,3\}$*, then functions in* $H^2(\Omega)$ *are continuous.*

4.2 Fractional Sobolev spaces

In the preceding sections, we have only looked at Sobolev space with $k \in \mathbb{N}_0$ so far. There are various approaches on how to define Sobolev spaces with fractional order. The approach that we will follow here mimics the idea of Hölder spaces. For Hölder spaces we know, for example, that the space $C^{0,\theta}$ with $\theta \in (0,1)$ contains functions that are somewhat more regular than just continuous functions, but provide less regularity than C^1 functions. In other words, $C^{0,\theta}(\Omega)$ lies between $C^0(\Omega)$ and $C^1(\Omega)$ (cf. interpolation spaces).

To carry the aforementioned intuition over, let $1 \le p < \infty$, $\theta \in (0,1)$, and $f \in L^p(\Omega)$ with Ω being an open subset of $\mathbb{R}^d$. First, we define the *Slobodeckij seminorm*

$$[f]_{\theta,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p + d}} \, dx \, dy \right)^{\frac{1}{p}} .$$

Next, for $0 < s \notin \mathbb{N}_0$, we set $\theta := s - \lfloor s \rfloor \in (0,1)$. Here, $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \colon m \le x\}$. Then the *Sobolev–Slobodeckij space* $W^{s,p}(\Omega)$ for the non-integer s is defined as

$$W^{s,p}(\Omega) := \left\{ f \in W^{\lfloor s \rfloor,p}(\Omega) \colon \sup_{|\boldsymbol{\alpha}| = \lfloor s \rfloor} [\partial^{\boldsymbol{\alpha}} f]_{\theta,p,\Omega} < \infty \right\} .$$

One can show that $W^{s,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{s,p}(\Omega)} := \|f\|_{W^{\lfloor s \rfloor,p}(\Omega)} + \sup_{|\boldsymbol{\alpha}| = \lfloor s \rfloor} [\partial^{\boldsymbol{\alpha}} f]_{\theta,p,\Omega} .$$

5 Trace Inequality and Trace space

From the considerations above, it is clear that if $v \in H^1(\Omega)$, we cannot define the "value" of v on the boundary $\partial\Omega$ of Ω (the so-called "trace") in a straightforward way, if the dimension is greater than one. Therefore, we need to introduce the following important result.

Theorem 5.1 (Trace theorem)**.** *Let* Ω *be a bounded open domain in* $\mathbb{R}^d$ *with Lipschitz boundary* $\partial\Omega$*. There exists a unique linear map*

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega),$$

such that $\gamma_0 v = v|_{\partial\Omega}$ *for all functions* $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$*. Here,* $\gamma_0 v$ *is called the* **trace** *of* v *on the boundary* $\partial\Omega$*. Moreover, there exists a positive constant* $C_T > 0$ *such that*

$$\|\gamma_0 v\|_{L^2(\partial\Omega)} \le C_T \|v\|_{H^1(\Omega)}.$$

Remark 5.1. *The trace of a non-continuous $H^1(\Omega)$ function is usually assigned by means of a "density argument". That is, although functions in $H^1(\Omega)$ are not necessarily continuous, it is possible to show that they can always be approximated accurately (with respect to the $H^1(\Omega)$ norm) by $C^\infty(\overline{\Omega})$ functions, provided that the boundary of Ω is sufficiently smooth (e.g. Lipschitz continuous), in the sense that*

$$\forall u \in H^1(\Omega),\ \exists \{v_n\}_{n\in\Omega} \in C^\infty(\overline{\Omega}) : \lim_{n\to\infty} \|u - v_n\|_{H^1(\Omega)} = 0 \,.$$

*In other words, $C^\infty(\overline{\Omega})$ is **dense** in $H^1(\Omega)$. As the functions v_n are continuous, their boundary value is given by $\gamma_0(v_n) = v_n|_{\partial\Omega}$ (see the Trace theorem just stated), and since the trace operator is continuous, the value of $\gamma_0 u$ is simply $\gamma_0 u = \lim_{n\to\infty} \gamma_0 v_n$. Observe that the trace $\gamma_0 u$ does not depend on the specific choice of the sequence $\{v_n\}_{n\in\mathbb{N}}$, i.e. any sequence $\{v_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ converging to $u \in H^1$ with respect to the H^1 norm will provide the same trace $\gamma_0 u$. Consequently, $\gamma_0 u$ is well defined.*

Remark 5.2. *The same result holds when considering the trace space over a Lipschitz continuous subset $\Gamma \subset \partial\Omega$ with positive measure.*

5.1 Trace space

The trace operator is not surjective on $L^2(\partial\Omega)$. In particular, the set of functions in $L^2(\partial\Omega)$ that are traces of functions of $H^1(\Omega)$ is a subspace of $L^2(\partial\Omega)$, which is denoted by $H^{1/2}(\partial\Omega)$. Similarly, for all $k \geq 1$, there exists a unique map

$$\gamma_{\partial\Omega} : H^k(\Omega) \to H^{k-1/2}(\partial\Omega)$$

such that $\gamma_{\partial\Omega} v = v|_{\partial\Omega}$ for all $v \in H^k(\Omega) \cap C^0(\bar{\Omega})$ with

$$\|\gamma_{\partial\Omega} u\|_{H^{k-1/2}(\partial\Omega)} \leq C_k^{\partial\Omega} \|u\|_{H^k(\Omega)}$$

for some positive constant $C_k^{\partial\Omega}$. Notice that we use the notation $\gamma_{\partial\Omega}$ instead of γ_0 here, in order to emphasise the dependence on the boundary of the domain (cf. $C_k^{\partial\Omega}$).

5.2 The space $H_0^1(\Omega)$

Let Ω be an open subset of $\mathbb{R}^d$. We define the space $H_0^1(\Omega)$ as

$$H_0^1(\Omega) = \left\{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \partial\Omega\right\},$$

where $v = 0$ on $\partial\Omega$ is to be interpreted in the trace sense $\gamma_0 v = 0$. A similar characterisation holds for the space

$$H_\Gamma^1(\Omega) = \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma\}, \quad \Gamma \subset \partial\Omega,\ meas(\Gamma) > 0 \,.$$

6 The Poincaré inequality

A list of very useful inequalities that relate the L^2 norm of an H^1 function to its H^1 seminorm are commonly referred to as Poincaré inequalities.

Theorem 6.1 (Poincaré inequality)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$. Then there exists a positive constant C_Ω such that*

$$\|v\|_{L^2(\Omega)} \leq C_\Omega |v|_{H^1(\Omega)} , \qquad \forall v \in H_0^1(\Omega) . \tag{2}$$

Proposition 6.1 (Friedrichs' inequality)**.** *The Poincaré inequality is still valid if the function v vanishes only on a part of the boundary $\Gamma \subset \partial\Omega$ with meas$(\Gamma) > 0$. In this case, it holds that*

$$\|v\|_{L^2(\Omega)} \leq C_\Omega^* |v|_{H^1(\Omega)} , \qquad \forall v \in H_\Gamma^1(\Omega) = \left\{ v \in H^1(\Omega) \, : \, v = 0 \text{ on } \Gamma \right\} .$$

Proposition 6.2 (Poincaré–Wirtinger inequality)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$. For all $v \in H^1(\Omega)$ with $\int_\Omega v(x)\,dx = 0$ there exists a positive constant C_Ω such that*

$$\|v\|_{L^2(\Omega)} \leq C_\Omega |v|_{H^1(\Omega)} . \tag{3}$$

7 Linear functionals

Let V and W be two linear spaces over $\mathbb{R}$. A map $L : V \to W$ is said to be **linear** if:

$$\begin{aligned} L(u+v) &= L(u) + L(v), \quad \forall u, v \in V, \\ L(\alpha v) &= \alpha L(v), \quad \forall \alpha \in \mathbb{R}. \end{aligned}$$

Let $(V, \|\cdot\|)$ be a normed space and $F : V \longrightarrow \mathbb{R}$ a linear map. We say that:

- F is **bounded**, if there exists a positive constant $C < \infty$ such that $|F(v)| \leq C\|v\|$ for all $v \in V$.
- F is **continuous**, if for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\|u - v\| < \delta_\varepsilon$, then $|F(u) - F(v)| < \varepsilon$.
- F is **Lipschitz continuous**, if $|F(v - w)| \leq M\|v - w\|, \quad M < \infty$.

A linear and continuous operator $F : V \longrightarrow \mathbb{R}$ is said to be a **linear functional**. Furthermore, the following result holds.

Lemma 7.1. *If $F : V \longrightarrow \mathbb{R}$ is linear, then F is (Lipschitz) continuous if and only if F is bounded.*

7.1 The dual space

We define the **dual space** of a normed space $(V, \|\cdot\|)$ as the space $(V', \|\cdot\|_{V'})$ with

$$\begin{aligned} V' &= \{F : V \to \mathbb{R} \, : \, F \text{ linear and continuous}\} , \\ \|F\|_{V'} &= \sup_{v \in V, v \neq 0} \frac{|F(v)|}{\|v\|}, \quad \forall F \in V'. \end{aligned}$$

It can be shown that $\|\cdot\|_{V'}$ is a norm on V'. In particular, the supremum exists as F is bounded. Thus, $(V', \|\cdot\|_{V'})$ is a bounded normed linear space. Moreover, it can be shown that V' is a Banach space (regardless of whether or not V is a Banach space).

7.2 The Riesz Representation Theorem

For Hilbert spaces, we have the following fundamental result concerning the representation of elements of the dual space.

Lemma 7.2 (Riesz representation theorem)**.** *Let H be a Hilbert space with norm $\|\cdot\|_H$ and scalar product $(\cdot,\cdot)_H$. For all $F \in H'$ (the dual space of H), there exists a unique $v \in H$ such that*

$$F(x) = (v, x)_H \ , \qquad \forall x \in H \ .$$

Moreover, $\|F\|_{H'} = \|v\|_H$.

7.3 Closed Range Theorem

Let us now consider a more general setting. To this end, let V and W be two Banach spaces. Then, consider a linear operator (i.e. a map) $L\colon D(L) \subset V \to W$, which is defined on a linear subspace $D(L) \subset V$ with values in W. The set $D(L)$ is called the *domain* of L. For such an operator L, we introduce:

- the *range* (or image) of L:

$$R(L) \equiv \mathrm{Im}(L) := \{Lv\colon v \in D(L)\} \subset W \ ,$$

- the *kernel* (or null space) of L:

$$N(L) \equiv \mathrm{Ker}(L) := \{v \in D(L)\colon Lv = 0\} \subset V \ ,$$

- the *graph* of L:

$$G(L) := \{(v, Lv)\colon v \in D(L)\} \subset V \times W \ .$$

A map L is said to be *closed*, if its graph $G(L)$ is closed in $V \times W$. To prove that an operator L is closed, one typically proceeds as follows. Consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subset D(L)$ with $v_n \to v$ in V and $Lv_n \to w$ in W. One then needs to check that both $v \in D(L)$ and $w = Lv$ hold.

Most practically relevant operators are indeed closed and also *densely defined*, in the sense that $D(L)$ is dense in V. For these operators, we introduce the *adjoint* $L^\star\colon D(L^\star) \subset W' \to V'$, which is the linear operator satisfying

$$F(Lv) = (A^\star F)(v) \ , \quad \forall v \in D(L) \ , \quad \forall F \in D(L^\star) \ .$$

The domain of the adjoint is

$$D(L^\star) := \{F \in W'\colon \exists c \geq 0 \text{ s.t. } |F(Lv)| \leq c\|v\| \ , \ \forall v \in D(L)\} \ ,$$

which is a linear subspace of W'. Now, we can state the main result concerning the range of L and $L^\star$.

Theorem 7.1 (Closed range theorem)**.** *Let $L\colon D(L) \subset V \to W$ be a linear operator that is closed and densely defined. Then the following properties are equivalent:*

1. *$R(L)$ is closed,*

2. $R(L^\star)$ *is closed,*

3. $R(L) = N(L^\star)^\perp \equiv \{w \in W\colon F(w) = 0 \ \ \forall F \in N(L^\star)\}$,

4. $R(L^\star) = N(L)^\perp \equiv \{F \in V'\colon F(v) = 0 \ \ \forall v \in N(L)\}$.

Notice the resemblance to the fundamental theorem of linear algebra.