Numerical Approximation of PDEs

Spring Semester 2025

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Session 09: May 1, 2025

Exercise 1. Let $\left(H^1_0(\Omega)\right)'$ be the space of functionals f over $H^1_0(\Omega)$, i.e., $\left(H^1_0(\Omega)\right)'$ contains all distributions f such that $\langle f, u\rangle \in \mathbb{R}$. Here $\langle \cdot\,,\,\cdot\rangle$ denotes the application of f to u. The space $\left(H^1_0(\Omega)\right)'$ is endowed with its natural norm

$$\|f\|_{\left(H^1_0(\Omega)\right)'} = \sup_{u\in H^1_0(\Omega)} \frac{\langle f, u\rangle}{\|u\|_{H^1_0(\Omega)}}.$$

Consider the weak formulation of the heat equation. Discuss, conceptually, why it makes sense to view $\partial_t u$ as a function

$$\partial_t u : [0, T] \to H^1_0(\Omega)'$$

from the time interval into the dual space.
Suppose that $u \in L^2(0, T; H^1_0(\Omega))$ is a Galerkin solution, and assume for simplicity that

$$u \in C(0, T; H^1_0(\Omega)), \quad \partial_t u \in C(0, T; H^1_0(\Omega)').$$

Prove that

$$\int_0^T \|\partial_t u\|^2_{H^1_0(\Omega)'} dt \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)} dt\right).$$

Exercise 2. Consider the heat equation over a domain Ω,

$$\partial_t u - \Delta u = 0$$

with homogeneous Dirichlet boundary conditions and with initial data $u_0 \in L^2(\Omega)$.
Suppose that $u_h : [0, T] \to V_h$ is a semidiscrete approximation with some Galerkin space V_h and initial data $u_{h,0} \in V_h$. Show that $\|u_h(t)\|_{L^2(\Omega)}$ is non-increasing in time.

Exercise 3. Prove the following identity: when $v_h = \theta u_h^{n+1} + (1-\theta)u_h^n$, then

$$\int_\Omega \left(u_h^{n+1} - u_h^n\right) v_h = \frac{1}{2}\|u_h^{n+1}\|^2_{L^2} - \frac{1}{2}\|u_h^n\|^2_{L^2} + (\theta - \frac{1}{2})\|u_h^{n+1} - u_h^n\|^2_{L^2}.$$

Exercise 4. Let $\Omega = [0, 1]^2$ and $T = 10$. Consider the heat equation with Neumann boundary conditions and vanishing source term:

$$\partial_t u - \operatorname{div}(\nabla u) = 0 \text{ over } \Omega \times (0, T)$$
$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_N} = 0 \text{ along } \partial\Omega$$
$$u(x, 0) = 1 + \sin(2\pi x)\sin(2\pi y) \text{ at } t = 0.$$

- Find the semidiscrete formulation using the piecewise linear finite element space.
- We are interested in the integral

$$H(t,u) := \int_\Omega u(x,t)dx.$$

  We know that $H(t,u)$ is constant in t for the solution u of the heat equation. Does $H(t,u_h)$ stay constant, where u_h is the solution of the semi-discrete formulation?

- Use the templates on GitHub (the link is on the Moodle page) to investigate computationally for which time discretizations this integral remains constant. Prove that it stays constant when using the explicit Euler and Crank-Nicolson method.
- A common difficulty with finite elements for the heat equation is that the time-discretization methods require solving a system with the mass matrix M. It is therefore common to replace the mass matrix by a diagonal matrix $\tilde{M}$, which is then called *lumped mass matrix*. One way of doing this to define the diagonal entries of $\tilde{M}$ as

$$\tilde{M}_{ii} = \sum_j M_{ij}.$$

  That is, we *lump* the entries on each row into one. Implement the explicit Euler and Crank-Nicolson scheme with the lumped mass matrix and assess how these compare to the original versions. Use conjuate gradient method to solve the linear system at each iteration. Compare the solution of the lumped scheme to the solution of the non-lumped scheme by computing the $L^2(\Omega)$-discrepancy between the two soltutions at select time instances.