Numerical Approximation of PDEs

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Exercise 1. [The reference element] Let K be a positively-oriented triangle. We consider a reference transformation $F_K : \hat{K} \to K$ with B_K as defined in the lecture.

(1a) Show that $\det B_K$ relates to the surface area $|K|$ of the triangle K as follows:

$$\det B_K = 2|K|.$$

(Note that there is a mistake in the lecture notes, erroneously claiming that $\det B_K = \frac{1}{2}|K|$*).*

(1b) Proof that the following two estimates hold

$$\|B_K\| \leq \frac{h_K}{\hat{\rho}}, \quad \left\|B_K^{-1}\right\| \leq \frac{\hat{h}}{\rho_K}, \tag{1}$$

where $\hat{\rho}$ and $\hat{h}$ are the inner and outer diameters of the reference triangle $\hat{K}$ while ρ_K and h_K denote the inner and outer diameters of the triangle K.

Deduce that there exists $C_0, C_1 > 0$ independent of K such that :

$$C_0 \rho_K \leq \|B_K\| \leq C_1 h_K, \quad \text{and} \quad C_2 \frac{\rho_K}{h_k} \leq \|B_K\| \|B_K^{-1}\| \leq C_3 \frac{h_K}{\rho_K}. \tag{2}$$

(1c) The quantity h_K/ρ_K is often called aspect ratio in the literature.[1] Someone proposes instead to measure the quality of a triangle by the ratio of the longest edge and the shortest edge. Is that a good idea?

Exercise 2. [Finite Element Method in 1D] Consider Poisson's equation with a non-constant diffusion coefficient $k(x) \in C^0([a, b])$, where $k(x) > 0$ for all $x \in (a, b)$:

$$\begin{cases} -(k(x)u'(x))' = f(x), & x \in (a, b), \\ u(a) = g_a, \quad u(b) = g_b. \end{cases} \tag{3}$$

(2a) Use the midpoint quadrature formula to derive the stiffness matrix A with entries

$$A_{i,j} = \int_a^b k(x)\phi_i'(x)\phi_j'(x)dx$$

and right hand side for a piecewise linear finite element approximation of the solution of system (3) with $g_a = g_b = 0$, when using a uniform grid, and compare it to the stiffness matrix

[1] Different authors use different definitions.

that arises when using a second-order accurate finite difference approximation.

(2b) Implement the finite element approximation derived in **(2a)** for $a = 0, b = 1, g_a = 0$ and $g_b = 0$. Here, the diffusion coefficient and the right hand side are defined as

$$k(x) = \begin{cases} 0.5 + x, & x \leq 1/2 \\ 1.5 - x, & x > 1/2 \end{cases}$$

$$f(x) = \begin{cases} 0.5\pi \sin(\pi x) + \pi x \sin(\pi x) - \cos(\pi x), & x \leq 1/2 \\ 1.5\pi \sin(\pi x) - \pi x \sin(\pi x) + \cos(\pi x), & x > 1/2. \end{cases}$$

Note that the exact solution is given by $u(x) = \frac{1}{\pi} \sin(\pi x)$.

(2c) Derive a piecewise linear finite element approximation of problem (3) with $k(x) = k =$ constant and non-homogeneous Dirichlet boundary conditions $u(a) = g_a$ and $u(b) = g_b$.
Hint: Write the solution as a linear combination of the basis functions $(\phi_0, \dots, \phi_N)$ and split

$$u_h = u_h^0 + g_h, \text{ where } u_h^0(x) \text{ is zero on the boundary, while } g_h = g_a\phi_0 + g_b\phi_N. \tag{4}$$

Note that this essentially eliminates two unknowns from the problem!