Numerical Approximation of PDEs

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Lecturer: Prof. Annalisa Buffa — Assistant: Mohamed Ben Abdelouahab

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Exercise 1. [The MINI Element for the steady Stokes problem]

- Consider the Stokes problem:

$$\begin{cases} -\Delta \underline{u} + \nabla p &= \underline{f} \quad \text{in } \Omega, \\ \nabla \cdot \underline{u} &= 0 \quad \text{in } \Omega, \\ \underline{u} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

  Write the variational formulation.

- If you replace the Hilbert spaces with finite dimensional spaces V_h for $\underline{u}$ and Q_h for p; then the discrete problem takes the form of the following linear system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

  To what corresponds the matrices A and B ?

- Let $\mathcal{T}_h$ be a conforming triangulation of $\Omega \subset \mathbb{R}^2$. For each triangle $T \in \mathcal{T}_h$, define the bubble function:

$$b_T(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x),$$

  where $\lambda_i(x)$ are the barycentric coordinates on T.

  Define the velocity and pressure spaces as:

$$V_h = \left\{ \underline{u}_h \in [H_0^1(\Omega)]^2 : \underline{u}_h|_T \in [\mathbb{P}^1(T) \oplus \operatorname{span}(b_T)]^2, \, \forall T \in \mathcal{T}_h \right\},$$
$$Q_h = \left\{ q_h \in L_0^2(\Omega) \cap C^0(\Omega) : q_h|_T \in \mathbb{P}^1(T), \, \forall T \in \mathcal{T}_h \right\}.$$

  Prove that with this choice of V_h and Q_h (known as the MINI Element), the matrix $B \in \mathbb{R}^{M \times N}$ has full rank.

  Note that: $\dim(V_h) = \#V_0 + \#T = N > \dim(Q_h) = \#V - 1 = M$, where

  - $\#V$ is the total number of vertices in $\mathcal{T}_h$.
  - $\#V_0$ is the number of interior vertices.
  - $\#T$ is the number of triangles in $\mathcal{T}_h$.

NB. For further details, see the book by Boffi, Brezzi and Fortin.