# Numerical Approximation of PDEs

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**Exercise 1.** We consider the wave equation

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \text{ in } \Omega \times (0, T], \\[2mm]
u = 0 \text{ on } \partial\Omega \times (0, T], \\[2mm]
u(0) = u_0 \text{ in } \Omega, \\[2mm]
\dfrac{\partial u}{\partial t}(0) = v_0 \text{ in } \Omega.
\end{cases}
$$

Given a uniform grid of $[0, T]$

$$
0 = t^0 < t^1 < ... < t^N = T,
$$

the fully discrete approximation of the wave equation with Finite Elements of degree $r$ and the explicit Newmark method to advance in time reads : starting from $u_h^0 = I_h^r u_0$ and $v_h^0 = I_h^r v_0$ solve for $n = 0, 1, 2, ..., N$

$$
\int_\Omega \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} v_h + c^2 \int_\Omega \nabla u_h^n \nabla v_h = 0, \forall v_h \in X_{h,0}^r, n \geq 1,
$$

with the initialization

$$
\int_\Omega \frac{u_h^1 - u_h^0}{\Delta t} v_h + \frac{c^2 \Delta t}{2} \int_\Omega \nabla u_h^0 \nabla v_h = \int_\Omega v_h^0 v_h, \forall v_h \in X_{h,0}^r.
$$

Prove that the Newmark method is stable under the CFL condition

$$
c\Delta t \leq \frac{1}{C_I} h
$$

where $C_I$ is the constant of the inverse inequality. In particular there exists a pure real constant $C > 0$ such that for all $m = 1, 2, ..., N$

$$
\left\| \frac{u_h^m - u_h^{m-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^m\|_{L^2(\Omega)}^2 \leq C \left( \|v_h^0\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^0\|_{L^2(\Omega)}^2 \right).
$$

*Hint 1:* For the first step, choose $v_h = \frac{u_h^1 - u_h^0}{\Delta t}$ and prove that under the CFL condition it holds

$$\left\| \frac{u_h^1 - u_h^0}{\Delta t} \right\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^1\|_{L^2(\Omega)}^2 \leq C \left( \|v_h^0\|_{L^2(\Omega)}^2 + c^2 \|\nabla u_h^0\|_{L^2(\Omega)}^2 \right)$$

where $C > 0$ is a pure real constant.

*Hint 2:* For $n \geq 1$, choose $v_h = \frac{u_h^{n+1} - u_h^n}{\Delta t} + \frac{u_h^n - u_h^{n-1}}{\Delta t}$ as for the proof in the implicit case, and observe that

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} + \frac{u_h^n - u_h^{n-1}}{\Delta t} = \frac{u_h^{n+1} - u_h^{n-1}}{\Delta t}.$$

Then note that

$$\int_\Omega \nabla u_h^n \nabla (u_h^{n+1} - u_h^{n-1}) = \int_\Omega \nabla u_h^{n+1} \nabla u_h^n - \int_\Omega \nabla u_h^n \nabla u_h^{n-1}$$

is a telescopic sum.