Chapter 1 Linear elliptic PDEs of 2nd order

$\boxed{-\Delta u = f}$ on Ω

$-\Delta = \begin{cases} -u'' & d=1 \\ -(u_{xx}+u_{yy}) & d=2 \\ \vdots & \end{cases}$

d = space dimension $\quad \Omega \subseteq \mathbb{R}^d$, Ω connected, smooth or polygonal, open.

$\boxed{-\operatorname{div}(\mathbb{K}\nabla u) = f}$

∇ = gradient

$\operatorname{div}(\underline{w}) = \frac{\partial}{\partial x} w_1 + \frac{\partial}{\partial y} w_2 + \dots$

$\mathbb{K}(\underline{x}) \in \mathbb{R}^{d\times d}$ symmetric and positive definite. (diffusion coefficient) for every $\underline{x} \in \Omega$

$\underline{d=1}$ $\qquad -(K u')' = f \qquad K(x) > 0$

1.2 Wellposedness for d = 1.

$\begin{cases} -(K u')' = f & \text{on } \Omega \\ \text{boundary conditions.} & \end{cases}$

$u : \Omega \to \mathbb{R}$

Dirichlet $\begin{cases} u(a) = g_a \\ u(b) = g_b \end{cases}$

Neumann $\begin{cases} u'(a) = g'_a \\ u'(b) = g'_b \end{cases}$

mixed $\begin{cases} u(a) = g_a \\ u'(b) = g'_b \end{cases}$ or the other way.

$$(1)\begin{cases} -(ku')' = f \\ u(a) = u(b) = 0 \end{cases}$$ homogeneous boundary conditions.

classical solutions : $f \in C^0(\Omega)$ $u \in C^2(\Omega)$?

for smooth coefficient $k(x)$.

seeking for a distributional solutions

$\varphi \in C_0^\infty(\Omega)$ $$\int_a^b (-ku')'\varphi = \int_a^b f\varphi \quad (\circ)$$

$$\int_a^b -(ku')'\varphi = \int_a^b ku'\varphi' + \cancel{ku'(a)\varphi(a)} - \cancel{ku'(b)\varphi(b)}$$

$\varphi(a) = \varphi(b) = 0.$

$(\circ)$ $\int_a^b ku'\varphi' = \int_a^b f\varphi$ $\forall \varphi \in C_0^\infty(\Omega)$ [C^∞ functions with compact support]

$\Omega = (a,b)$ or $\Omega \subset \mathbb{R}^d$ open regular

$\boxed{L^2(\Omega)} = \{ u : \Omega \to \mathbb{R} \ \int_\Omega |u|^2 < \infty \}.$

$$\|u\|^2_{L^2(\Omega)} = \|u\|_0^2 = \int_\Omega |u|^2$$

$\boxed{H^1(\Omega)} = \{ u \in L^2(\Omega) : u' \in L^2(\Omega) \}.$

[$H^1(\Omega) = \{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega)^d \}.$]

$$\|u\|^2_{H^1} = \|u\|^2_1 = \|u\|_0^2 + \|\nabla u\|_0^2$$

theorem: for $d = 1$ $H^1(\Omega) \hookrightarrow C^0(\Omega)$

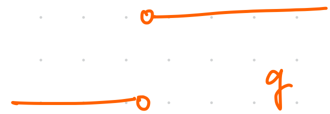
then, I can set Dirichlet boundary condition of every u in $H^1(\Omega)$

$$H^1_0(\Omega) = \{ u \in H^1(\Omega) \;:\; u(a) = u(b) = 0 \}.$$

$$\int_a^b k u' \varphi' = \int_a^b f \varphi \qquad \varphi \in C_0^\infty(\Omega)$$

[distributional derivatives g distribution $g \in \mathcal{D}'(\Omega)$

$$\langle g', \varphi \rangle = - \langle g, \varphi' \rangle$$



I will provide a reference for it]

Given $f \in L^2(\Omega)$ $k > 0$, $k \in \mathbb{R}$

$$\boxed{\begin{array}{l}\text{Find } u \in H^1_0(\Omega) \text{ such that} \\ \displaystyle\int_a^b k u' \varphi' = \int_a^b f \varphi \qquad \forall \varphi \in H^1_0(\Omega)\end{array}} \qquad (2)$$

variational formulation of the problem. (1)

theorem For any $f \in L^2(\Omega)$ k, $k(x) > 0$ and bounded there exists a unique solution $u \in H^1_0(\Omega)$ of problem (2). and moreover $\|u\|_1 \leq C \|f\|_0$

exercise: prove that if u is regular enough then, u is also a solution of (1) pointwise.

1.3 Lax-Milgram lemma

- Given a Hilbert space V
- Given a bilinear form on V

$a(\cdot,\cdot)$ $\qquad a(u,v) \in \mathbb{R}$

$u, v \in V.$

such that $\exists M, \alpha > 0$

$a(u,v) \leq M \|u\|_V \cdot \|v\|_V$

continuity.

$a(u,u) \geq \alpha \|u\|_V^2$

coercivity

then, given a functional F on V, the problem

$a(u,v) = F(v)$ admits

a unique solution.

Moreover $\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$

$H_0^1(\Omega)$

$a(u,\varphi) = \int_\Omega k u' \varphi'$

$\int_\Omega k u' \varphi' \overset{?}{\leq} M \|u\|_1 \cdot \|\varphi\|_1$

$\int_\Omega k |u'|^2 \overset{?}{\geq} \alpha \|u\|_1^2$

$F = f$

$F(\varphi) = \int_\Omega f\varphi$

$\int_\Omega k u' \varphi' = \int_\Omega f\varphi$

admits a unique solution.

$\|F\|_{V'} \rightarrow \|f\|_0$

- continuity $\qquad \int_a^b k u' \varphi' \leq M \|u\|_1 \cdot \|\varphi\|_1$

Cauchy-Schwarz inequality: $f, g \in L^2(\Omega)$ then

$$\int_\Omega f g \leq \|f\|_0 \, \|g\|_0 .$$

$$\left(\int_a^b |f|^2\right)^{1/2} \cdot \left(\int_a^b |g|^2\right)^{1/2}$$

k bounded $\qquad \int_\Omega k\, f g \leq \underset{\displaystyle \max_x |k(x)|}{C(k)}\, \|f\|_0 \cdot \|g\|_0$

$$\int_a^b k\, u'\varphi' \overset{C.S.}{\leq} C(k) \cdot \|u'\|_0 \cdot \|\varphi'\|_0 \leq (\cdot)$$

remember $\quad \|u\|_1^2 = \|u\|_0^2 + \|u'\|_0^2$

then $\quad \|u'\|_0 \leq \|u\|_1 \quad$ by construction.

$$(\cdot) \leq \underbrace{C(k)}_{M} \|u\|_1 \cdot \|\varphi\|_1$$

continuity is true!

<u>Coercivity</u>

$$a(u,u) \geq \alpha \|u\|_V^2$$

$$\int k|u'|^2 \overset{?}{\geq} \alpha \|u\|_1^2$$

: to be proved

$$\int_a^b k\,|u'|^2 \geq \left[\min_{x\in[a,b]} k(x)\right] \int_a^b |u'|^2$$

$$\geq \left[\min_{x\in[a,b]} k(x)\right] \|u'\|_0^2$$

I can conclude only if I can prove that $\quad \|u\|_0 \leq C_p \|u'\|_0$

Poincaré inequality

Poincaré inequality. $u \in H^1_0(\Omega)$ it holds

$$\|u\|_0 \leq C_p \|u'\|_0.$$

Let's use this one for coercity.

$$\begin{aligned}
\int k|u'|^2 &\geq C_k \|u'\|_0^2 \\
&\geq C_k \left(\frac{1}{2}\|u'\|_0^2 + \frac{1}{2}\|u'\|_0^2 \right) \\
&\geq C_k \left(\frac{1}{2}\|u'\|_0^2 + \frac{1}{2} \cdot \frac{1}{C_p^2} \cdot \|u\|_0^2 \right) \\
&\geq \underbrace{C_k \cdot \min\left[\frac{1}{2}, \frac{1}{2C_p^2}\right]}_{\alpha} \cdot \|u\|_1^2
\end{aligned}$$

Coercity is proved!