$$\int_\Omega \frac{\partial u_N}{\partial t} v_N + \int_\Omega \underset{\text{scalar}}{A} \nabla u_N \nabla v_N = \int_\Omega f v_N \qquad \forall v_N \in V_N$$

$$\underline{\underline{M}} \dot{u}_N + \underline{\underline{A}} u_N = F_N \qquad \begin{aligned} M &= \text{mass} \\ A &= \text{stiffness.} \end{aligned}$$

M invertible
A symm. pos. def. $\Rightarrow \exists!\, u_N$.

$v_N = u_N$

$$\int_0^T \frac{1}{2} \int_\Omega \frac{\partial}{\partial t} |u_N|^2 + \int_0^T \int_\Omega A |\nabla u_N|^2 = \int_0^T \int_\Omega f u_N$$

$$\frac{1}{2} \int_\Omega |u_N|^2(T) - \frac{1}{2} \int_\Omega |u_N|^2(0) + \underbrace{\int_0^T \int_\Omega A |\nabla u|^2}_{} = \int_0^T \int_\Omega f u_N.$$

Poincaré inequality. $\quad \|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2}$

$$\Rightarrow \|u\|_{H^1} \leq C' \|\nabla u\|_{L^2}$$

$$\Rightarrow \int_0^T \int_\Omega A |\nabla u|^2 \geq \underbrace{\alpha \int_0^T \|u\|^2_{H^1}}_{}$$

$$\underset{\underset{L^\infty(0,T;L^2)}{\wedge\!\!|}}{\frac{1}{2} \|u_N(T)\|^2_{L^2}} + \alpha \underbrace{\int_0^T \|u_N\|^2_{H^1}}_{L^2(0,T;H^1)} \leq \underbrace{\int_0^T \int_\Omega f u_N + \frac{1}{2} \|u_N(0)\|^2_{L^2}}_{\odot}$$

$$\odot \int_0^T \int_\Omega f u_N \leq \int_0^T \|f\|_{L^2} \|u_N\|_{L^2} \leq \|f\|_{L^2(0,T;L^2)} \cdot \|u_N\|_{L^2(0,T;L^2)}$$

$2ab \leq a^2 + b^2$ $\qquad$ $a^2 + b^2 - 2ab = (a-b)^2 \geq 0.$

$$ab = (\beta a)\left(\frac{1}{\beta} b\right) \leq \frac{\beta^2 a^2}{2} + \frac{1}{2\beta^2} b^2$$

$\Longrightarrow$

$$\int_0^T \int_\Omega f u_N \leq \frac{1}{2\alpha} \|f\|^2_{L^2(0,T;L^2)} + \frac{1}{2}\alpha \|u_N\|^2_{L^2(0,T;H^1)}$$

$$\|u_N\|_{L^2(L^2)} \leq \|u_N\|_{L^2(H^1)}$$

$$\frac{1}{2}\|u_N(T)\|^2_{L^2} + \alpha \|u_N\|^2_{L^2(H^1)} \leq \frac{1}{2\alpha}\|f\|^2_{L^2(L^2)} + \boxed{\frac{1}{2}\alpha \|u_N\|^2_{L^2(H^1)}} + \|u_N(0)\|^2_{L^2}$$

$$\frac{1}{2}\|u_N(t)\|^2_{L^2} + \frac{1}{2}\alpha \|u_N\|^2_{L^2(H^1)} \leq \frac{1}{2\alpha}\|f\|^2_{L^2(L^2)} + \|u_N^0\|^2_{L^2}$$

initial condition

$\hookrightarrow$ u_N is stable in $L^\infty(0,T;L^2)$ and $L^2(0,T;H^1(\Omega))$

$\exists$ subsequence of u_N : $u_N \longrightarrow u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$

and "u is now solution of the original problem."

$$\frac{\partial u_N}{\partial t} \xrightarrow{\text{weak conv.}} \frac{\partial u}{\partial t} \qquad \nabla u_N \xrightarrow{\text{weak conv.}} \nabla u$$

this remains to be checked.

$v_N = \dfrac{\partial u_N}{\partial t}$ my $\overset{\text{new}}{\vee}$ choice of test function.

$$\int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \int_0^T \int_\Omega A \underbrace{\nabla u_N \nabla \frac{\partial u_N}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}|\nabla u_N|^2} = \int_0^T \int f \frac{\partial}{\partial t} u_N$$

$$\left[\ \left\| \frac{\partial}{\partial t} u_N \right\|^2_{L^2(L^2)} + \frac{1}{2} \int_\Omega A |\nabla u_N(T)|^2 + \right.$$

$$\left. - \frac{1}{2} \int_\Omega A |\nabla u_N(0)|^2 \ \right] = \underbrace{\int_0^T \int_\Omega f \frac{\partial u_N}{\partial t}}$$

$$\underbrace{\left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(L^2)} + \frac{1}{2} \alpha \|u_N(T)\|^2_{H^1}} \leq$$

$$\left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(L^2)} + \frac{1}{2} \int_\Omega A |\nabla u_N(T)|^2 =$$

$$= \underbrace{\int_0^T \int_\Omega f \frac{\partial u_N}{\partial t} + \frac{1}{2} \int_\Omega A |\nabla u_N(0)|^2} \leq$$

$$\leq \|f\|_{L^2(L^2)} \cdot \left\|\frac{\partial u_N}{\partial t}\right\|_{L^2(L^2)} \qquad \leq c \|u_N^0\|_{H^1}$$

$$\leq \frac{1}{2} \|f\|^2_{L^2(L^2)} + \boxed{\frac{1}{2} \left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(L^2)}} + c \|u_N^0\|_{H^1(\Omega)}$$

$$\frac{1}{2} \left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(L^2)} + \frac{1}{2} \alpha \|u_N(T)\|^2_{H^1} \leq$$

$$\leq \frac{1}{2} \|f\|^2_{L^2(L^2)} + c \|u_N^0\|_{H^1(\Omega)}$$

this provide the following
if $f \in L^2(L^2)$, $u_N^0 \in H^1(\Omega)$ $\Rightarrow$

$$\frac{\partial u_N}{\partial t} \in L^2(0,T; L^2) \qquad u_N \in L^\infty(0,T; H^1)$$

$$\frac{\partial u_N}{\partial t} \rightarrow \frac{\partial u}{\partial t} \qquad u_N \rightarrow u \text{ in } L^\infty(H^1)$$

$\Rightarrow$ the solution of the heat equation exists and verifies the regularity expressed in the theorems. ∎

4.2 Finite elements for the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) = f & \Omega \longleftrightarrow T_h \\ u(x,0) = u_0(x) & \Omega \\ \gamma_D(u(x,t)) = g(x,t) & \partial\Omega \qquad g=0 \end{cases}$$

Finite element in space + finites difference in time

$\underset{\uparrow}{\text{Finite element}}$: $\mathbb{P}^1, \dots, \mathbb{P}^r$

finites difference:
- implicit / explicit Euler
- multistep
- Runge-Kutta...

$$V_h = \{ u_h \in H_0^1(\Omega) \; : \; u_h|_K \in \mathbb{P}^r(K) \quad \forall K \in T_h \}$$

Semi-discrete problem (continuous in time / discrete in space).

Find $u_h \in V_h$: $\forall v_h \in V_h$

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h + \int_\Omega A \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h$$

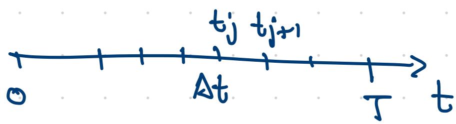
- $\|u_h(T)\|^2_{L^2} + \alpha \int_0^T \|u_h\|^2_{H^1} \le C_1 \|f\|^2_{L^2(L^2)} + C_2 \|u_0\|^2_{L^2}$
- $\|\frac{\partial}{\partial t} u_h\|^2_{L^2(L^2)} + \alpha \|u_h(T)\|^2_{H^1} \le C_1 \|f\|^2_{L^2(L^2)} + C_2 \|u_0\|^2_{H^1}$

$$\underline{\underline{M}}\, \dot{\underline{U}} + \underline{\underline{A}}\, \underline{U} = \underline{F}$$
$$\underline{U}(0) = \underline{U}_0$$

M = mass

A = stiffness.

Fully discrete problem



$$\underline{U}(t_{j+1}) = \underline{U}(t_j) + \int_{t_j}^{t_{j+1}} \dot{\underline{U}}(s)\, ds$$

$$M\, \underline{U}(t_{j+1}) = M\, \underline{U}(t_j) + \int_{t_j}^{t_{j+1}} M\, \dot{\underline{U}}(s)\, ds.$$
$$= M\, \underline{U}(t_j) + \int_{t_j}^{t_{j+1}} \left(-\underline{\underline{A}}\, \underline{U}(s) + F(s)\right) ds.$$

I apply the θ- method to the

$$M\, \underline{U}(t_{j+1}) = M\, \underline{U}(t_j) + \Delta t \Big[\theta \left(-\underline{\underline{A}}\, (\underline{U}(t_{j+1}) + \underline{F}_{j+1}\right)$$
$$+ (1-\theta)\left(-A\, \underline{U}(t_j) + F_j\right)\Big]$$

$\theta = 0.$ Explicit Euler.

$$\underline{M}\, \underline{U}(t_{j+1}) = M\underline{U}(t_j) - \Delta t\, \underline{\underline{A}} U(t_j) + \Delta t\, F_j$$

$\theta = 1$ Implicit Euler

$$\underline{M}\, \underline{U}(t_{j+1}) + \Delta t\, A U(t_{j+1}) = M U(t_j) + F_{j+1}$$

$$(\underline{M} + \Delta t\, \underline{\underline{A}})\, U(t_{j+1}) = \cdots$$

$\theta = 1/2$ Cranck Nickolson method (implicit)

Stability of the fully discrete problem

$u_h(x,t)$ Galerkin of semidiscrete problem

$u_h(x,t) = \sum_\ell U_\ell(t)\varphi_\ell(x)$ $\quad \varphi_\ell$ basis functions in space.

$\underline{U}(t)$ that solves $M\dot{U} + AU = F$

$$u_h^j(x) \simeq \sum_\ell U_\ell(t_j)\varphi_\ell(x)$$

$$\underbrace{\int_\Omega \frac{u_h^{j+1} - u_h^j}{\Delta t} \cdot v_h}_{\frac{MU^{j+1} - MU^j}{\Delta t}} + \int_\Omega \nabla(\theta u_h^{j+1} + (1-\theta)u_h^j)\nabla v_h = \int_\Omega (\theta f^{j+1} + (1-\theta) f^j)\, v_h.$$

Choosing a specific test function

$$v_h = \theta u_h^{j+1} + (1-\theta) u_h^j$$

Algebraic identity

$$\int_\Omega \left(u_h^{j+1} - u_h^j\right)\left(\theta u_h^{j+1} + (1-\theta) u_h^j\right) =$$

$$= \frac{1}{2} \|u_h^{j+1}\|_{L^2}^2 - \frac{1}{2} \|u_h^j\|_{L^2}^2 + (\theta - 1/2) \|u_h^{j+1} - u_h^j\|_{L^2}^2$$