3.2 Finite element approximation

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u) + \underline{b} \cdot \nabla u = f & \Omega. \\ \gamma(u) = g. & \partial\Omega \end{cases}$$

$u = u_0 + u_g$ $\qquad u_0 \in H^1_0(\Omega)$ $\qquad u_g$ lifting.

$V_h^g = \{ u \in H^1(\Omega) \quad \gamma(u) = I_h g \quad u|_K \in \mathbb{P}^r(K) \quad \forall K \in \mathcal{T}_h \}$

$$\|u - u_h\|_{H^1} \leq \underbrace{M/\alpha}_{} \inf_{v_h \in V_h^g} \|u - v_h\|_{H^1}$$

$$\frac{M}{\alpha} \sim Pe = \frac{|\underline{b}|}{\varepsilon}$$

$$r = 1 \quad \inf_{v_h} \|u - v_h\|_{H^1} \lesssim \underline{\underline{h}} . \cancel{\|u\|}_{H^2}$$

if $\quad Peh = \dfrac{|\underline{b}| h}{2\varepsilon} > 1 \quad \Rightarrow \quad Pe > \dfrac{2}{h}$

$\|u - u_h\|_{H^1} \lesssim Pe \cdot h \quad \Rightarrow$ the error is bounded by a constant that is not small.

Artifical viscosity

$$a(u,v) = \int_\Omega \varepsilon \nabla u \cdot \nabla v + (\underline{b} \cdot \nabla u) v$$

$$a_h(u,v) = \int_\Omega \left(\varepsilon + \gamma \frac{|\underline{b}| h}{2}\right) \nabla u \cdot \nabla v + (\underline{b} \cdot \nabla u) v$$

γ = a parameter that I need to choose.

let's try an error estimate:

$$a(u,v) = \int f v \qquad\qquad a_h(u_h; v_h) = \int_\Omega f v_h$$

- non-consistent method $\quad a_h(u, v_h) \neq \int_\Omega f v_h$

$a_h \qquad a_h(u,u) \geq \alpha_h \|u\|^2_{H^1}$

$\qquad\quad a_h(u,v) \leq M_h \|u\|_{H^1} \cdot \|v\|_{H^1}$

$$\alpha_h \|u_h - w_h\|^2_{H^1} \leq a_h(u_h - w_h, u_h - w_h) =$$

$$\underbrace{a_h(u_h - u, u_h - w_h)}_{\text{I}} + \underbrace{a(u - w_n, u_h - w_n)}_{\text{II}}$$

$$\text{II} \leq M_h \cdot \|u - w_n\|_{H^1} \|u_h - w_n\|_{H^1}$$

continuity of a_n

$$\text{I} = a_h(u_n, u_h - w_n) - a_h(u, u_h - w_n) = \int_\Omega f(u_h - w_n) - a_h(u, u_h - w_h)$$

$$= \cancel{\int_\Omega f(u_h - w_n)} - \cancel{\int \varepsilon \nabla u \nabla(u_h - w_n)} - \gamma \frac{|b| h}{2} \int \nabla u \nabla(u_h - w_n)$$

$$- \cancel{\int_\Omega b \cdot \nabla u (u_h - w_n)}$$

u = solution of the continuous problem!

$$\text{I} = -\frac{\gamma |b| h}{2} \int_\Omega \nabla u \nabla(u_h - w_n)$$

$$\text{I} \leq \|u_h - w_n\|_{H^1} \cdot \sup_{v_h \in V_h^0} \frac{-\frac{\gamma |b| h}{2} \int_\Omega \nabla u \cdot \nabla v_h}{\|v_h\|_{H^1}}$$

$$\leq \|u_h - w_n\|_{H^1} \frac{\gamma |b| h}{2} \cdot \|u\|_{H^1}$$

$$\alpha_h \|u_h - w_h\|^{\cancel{2}}_{H^1} \leq \quad I + II$$

$$\leq \gamma \frac{|b| h}{2} \|u\|_{H^1} \cdot \cancel{\|u_h - w_h\|} + M_h \|u - w_h\| \cdot \cancel{\|u_h - w_h\|}$$

where $w_h \in V_h^g$ is any function in the space.

$$\|u - u_h\|_{H^1} \leq \|u - w_h\|_{H^1} + \|u_h - w_h\|_{H^1}$$

$$\|u - u_h\|_{H^1} \leq \left(1 + \frac{M_h}{\alpha_h}\right) \|u - v_h\|_{H^1} + \frac{\gamma |b| h}{2} \|u\|_{H^1}$$

$M_h \lesssim |b|$.

$$\frac{M_h}{\alpha_h} = \frac{2|b|}{2\varepsilon + \gamma |b| h} \lesssim \frac{|b|}{\gamma |b| h} \lesssim \frac{1}{\gamma} \cdot \frac{1}{h}$$

$$\alpha_h = \varepsilon + \gamma \frac{|b| h}{2}$$

$$\|u - u_h\| \lesssim \left(1 + \frac{1}{\gamma} \cdot \frac{1}{h}\right) \inf_{w_h \in V_h^g} \|u - w_h\|_{H^1} + \frac{1}{\alpha} \frac{\gamma |b| h}{2} \cdot \|u\|_{H^1}$$

$\simeq 1$

Streamline diffusion

$$a_h(u, v) = a(u, v) + \frac{\gamma h}{|b|} \int_\Omega (b \cdot \nabla u)(b \cdot \nabla v)$$

Fully consistent methods

SUPG = streamline - Petrov Galerkin
GALS = Galerkin- least squares-

$$-\varepsilon \Delta u + b \cdot \nabla u = f \qquad \cdot\ b \cdot \nabla v$$

$$\int (-\varepsilon \Delta u + b \cdot \nabla u)\ b \cdot \nabla v = \int f\ b \cdot \nabla v$$

formally.

SUPG

$$a_h(u_h, v_h) = a(u_h, v_h) + \sum_{K \in \mathcal{T}_h} \delta_K \int_K (-\varepsilon \Delta u_h + b \cdot \nabla u_h) \cdot b \cdot \nabla v_h$$

$$\langle f_h, v_h \rangle = \int_\Omega f \cdot v_h + \sum_{K \in \mathcal{T}_h} \delta_K \int_K f \, \underline{b} \cdot \nabla v_h$$

SUPG is $\quad a_h(u_h, v_h) = \langle f_h, v_h \rangle$.

- SUPG is consistent

$$a_h(u, v_h) = \boxed{a(u, v_h)} + \sum \delta_K \int_K \boxed{\left(-\varepsilon \Delta u + \underline{b} \cdot \nabla u\right)} \cdot \underline{b} \nabla v_h$$

$$= \boxed{\int f v_h} + \sum \delta_K \int_K \boxed{f} \, \underline{b} \cdot \nabla v_h$$

Consistency $\rightarrow$ Galerkin orthogonality. $\rightarrow$

$$\| u - u_h \|_{H^1} \leq \text{"} M / \alpha \text{"} \, \| u - I_h u \|_{H^1}$$

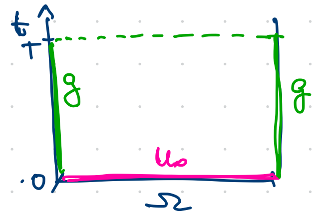
$a_h(\cdot, \cdot)$

$$\delta_K = \frac{h_K}{\|\underline{b}\|} \cdot \gamma_K \qquad \gamma_K = \text{depends on the degree,}$$

on the shape of K.

CHAPTER 4 Parabolic problems - Heat equation.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) = f & \Omega \times (0,T) \\ u(x,0) = u_0(x) & \Omega \\ \gamma(u(x,t)) = g(t) & \partial\Omega \end{cases}$$



propagation of the heat in a medium.

$u(x,t) : \Omega \times (0,T) \to \mathbb{R}$

$\underline{A}$ is the diffusion matrix, A is positive definite.

4.1 Weak formulation & wellposedness.

$$L^2(0,T; L^2(\Omega)) = \{ t \mapsto u(x,t) \in L^2(\Omega) \quad t \mapsto \|u\|_{L^2(\Omega)} \in L^2(0,T) \}$$

$$= \{ u(x,t) \quad \int_0^T \int_\Omega |u|^2 < +\infty \}.$$

$$L^2(0,T; H^1(\Omega)) = \{ t \mapsto u(x,t) \in H^1(\Omega) \quad t \mapsto \|u(x,t)\|_{L^2(\Omega)} \in L^2(0,T) \}.$$

$$= \{ u(x,t) : \int_0^T \|u\|^2_{H^1(\Omega)} < +\infty \}.$$

$$L^\infty(0,T; L^2(\Omega)) = \{ u(x,t) : \max_{\text{a.e. } t\in(0,T)} \|u\|_{L^2(\Omega)} < +\infty \}.$$

$$C^0(0,T; L^2(\Omega)) = \{ u(x,t) : \max_{t\in(0,T)} \|u\|_{L^2(\Omega)} < +\infty \}.$$

<u>theorem 1</u> If $f \in L^2(0,T; L^2(\Omega))$, $g = 0$

and $u_0 \in L^2(\Omega)$, then $\exists!\, u$

$u \in L^2(0,T; H^1(\Omega))$ [and $\dfrac{\partial u}{\partial t} \in L^2(0,T, (H^1_0(\Omega))')$]

theorem 2   Indeed $u \in C^0(0,T; L^2(\Omega))$

theorem 3   .. $f \in L^2(0,T; L^2(\Omega))$, $g = 0$ and

$u_0 \in H^1_0(\Omega) \implies$

$u \in L^\infty(0,T; H^1(\Omega))$ $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$

Sketch of the proofs of theorem 1 & 3.

$H^1_0(\Omega)$ $V_N = \text{span}\{\varphi_1 \dots \varphi_N\}$

$\{\varphi_1, \varphi_2, \dots\}$. is basis for $H^1_0(\Omega)$.

$$\int_\Omega \text{"} \frac{\partial u}{\partial t} - \text{div}(A \nabla u) = f * v \text{"}, \quad v \in H^1_0(\Omega)$$

$$\boxed{\int_\Omega \frac{\partial u}{\partial t} \cdot v + \int_\Omega A \nabla u \cdot \nabla v = \int_\Omega f v. \qquad v \in H^1_0(\Omega).} \quad (1)$$

Galerkin method $u_N \in V_N$ such that

$$\int_\Omega \frac{\partial u_N}{\partial t} v_N + \int_\Omega A \nabla u_N \cdot \nabla v_N = \int f v_N$$

$$u_N = \sum_i u_N^i(t) \varphi_i \quad \underline{U}_N = [u_N^1 \dots u_N^N] \qquad v_N = \sum v_N^e \varphi_e$$

test function.

$$\frac{\partial u_N}{\partial t} = \sum_i (u_N^i(t))' \varphi_i$$

$$\int_\Omega \frac{\partial u_N}{\partial t} \cdot v_N = \sum_i \sum_e (u_N^i(t))' v_e \overbrace{\int_\Omega \varphi_i \varphi_e}^{M_{ie}} = \underline{V}^T M \underline{\dot{U}}_N$$

$U_N = [u_N^1 \cdots \quad u_N^N]$ $\qquad\qquad$ $\underline{V} = [v_N^1 \cdots \quad v_N^N]$

$$\begin{cases} M\dot{\underline{U}}_N + A\underline{U}_N = \underline{f}_N \\ \underline{U}_N(0) = \underline{U}_N^0 \end{cases}$$

$f_N^i = \int f \varphi_i$

$\underline{U}_N^0$ = coefficients of $u_0(x)$ in the basis.

system of ODEs.

M is symmetric and positive definite
A is positive definite. $\quad \mapsto \exists! \, \underline{U}_N$

$\Rightarrow \quad u_N(x,t)$ exists as Galerkin solution of (1)

the solution u of (HE) is the "limit" of u_N.

We go back to Galerkin

$$\int_\Omega \frac{\partial u_N}{\partial t} v_N + \int_\Omega A \nabla u_N \nabla v_N = \int f v_N. \qquad v_N = u_N$$

$$\int_0^T \left[\frac{1}{2} \int_\Omega \frac{\partial}{\partial t} |u_N|^2 + \int_\Omega A |\nabla u_N|^2 = \int f u_N. \right]$$

$$\frac{1}{2} \|u_N(T)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_N(0)\|_{L^2(\Omega)} + \int_0^T \int_\Omega A |\nabla u_N|^2$$
$$= \int_0^T \int_\Omega f \cdot u_N.$$

- coercivity $\quad \int_0^T \alpha \|u_N\|^2_{H_0^1(\Omega)} \leq \int_0^T \int_\Omega A |\nabla u|^2$

$$\frac{1}{2} \|u_N(T)\|^2_{L^2(\Omega)} + \alpha \int_0^T \|u_N\|^2_{H_0^1(\Omega)} \leq \int_0^T \int_\Omega f u_N + \frac{1}{2} \|u_N(0)\|_{L^2(\Omega)}$$

the sequence u_N is bounded in $L^\infty(0,T;L^2(\Omega))$
and in $L^2(0,T;H^1_0(\Omega))$ ...

$$\int_0^T\int_\Omega f u_N \leq \int_0^T \|f\|_{L^2(\Omega)} \cdot \|u_N\|_{L^2(\Omega)}$$
$$\leq \|f\|_{L^2(L^2)} \cdot \|u_N\|_{L^2(L^2)}$$
$$\leq \|f\|_{L^2(L^2)} \cdot \|u_N\|_{L^2(H^1_0)}$$
$$\leq \frac{1}{2\alpha}\|f\|^2_{L^2(L^2)} + \frac{\alpha}{2}\|u_N\|^2_{L^2(H^1_0)}$$

$$2ab \leq a^2+b^2 \Longleftarrow (a-b)^2 = a^2+b^2-2ab \geq 0$$
$$2(\gamma a)\left(\frac{b}{\gamma}\right) \leq \gamma^2 a^2 + \frac{b^2}{\gamma^2}$$

$$\frac{1}{2}\|u_N(T)\|^2_{L^2(\Omega)} + \alpha\int_0^T \|u_N\|^2_{H^1_0(\Omega)} \leq \int_0^T\int_\Omega f u_N + \frac{1}{2}\|u_N(0)\|^2_{L^2(\Omega)}$$

$$\boxed{\frac{1}{2}\|u_N(T)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^T \|u_N\|^2_{H^1_0} \leq \frac{1}{2\alpha}\int_0^T \|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_N(0)\|^2_{L^2}}$$

(2)

initial condition

u_N is bounded

- there is a mathematical theory that guarantees that a subsequence of u_N admits a limit u

- such a limit is a solution of the heat equation.

Let's continue with the bounds.

$$\int_\Omega \frac{\partial u_N}{\partial t} \cdot v_N + \int_\Omega A \nabla u_N \cdot \nabla v_N = \int f v_N \qquad v_N = \frac{\partial u_N}{\partial t}$$

$$\int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \underbrace{\int_\Omega A \nabla u_N \nabla \frac{\partial u_N}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega A|\nabla u_N|^2} = \int f \frac{\partial u_N}{\partial t}$$

- $$\int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \frac{1}{2}\int_\Omega A|\nabla u_N(t)|^2 - \frac{1}{2}\int_\Omega A|\nabla u_N(0)|^2 = \int_0^T \int_\Omega f \frac{\partial u_N}{\partial t}.$$

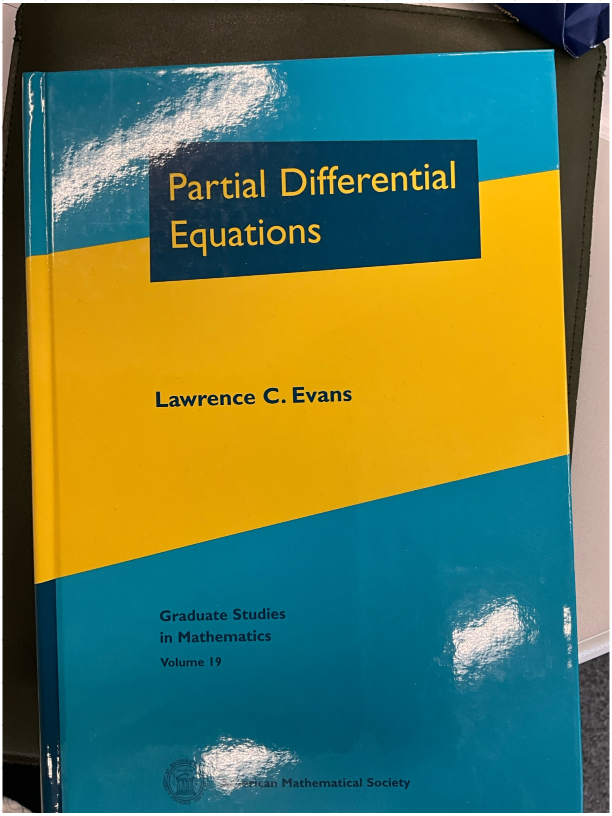
- $$\int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \frac{1}{2}\alpha \, \|u_N(t)\|^2_{H^1_0} \leq \|f\|_{L^2(L^2)} \cdot \left\|\frac{\partial u_N}{\partial t}\right\|_{L^2(L^2)} + \frac{1}{2} M \cdot \|u_N(0)\|^2_{H^1_0}.$$

- $$\boxed{\int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \alpha \, \|u_N(t)\|_{H^1_0} \leq \|f\|_{L^2(L^2)} + M \, \|u_N(0)\|_{H^1_0}} \quad (3)$$

if $u_N(0) \in H^1_0(\Omega)$, $u_N(0)$ is u_0, the initial cond.

$$\Rightarrow \quad \frac{\partial u_N}{\partial t} \in L^2(0,T; L^2(\Omega)), \qquad u_N \in L^\infty(0,T; H^1_0(\Omega))$$

$\rightarrow$ these regularities extend to the limit, i.e., to the solution u of (HE). ∎



Read here if you are interested in the fully detailed math. proof!

CHAPTER 7 :-)