$$\|u-u_h\|_{H^m(\Omega)} \leq C_I \frac{M}{\alpha} \quad h^{s-m} \quad |u|_{H^s(\Omega)}$$

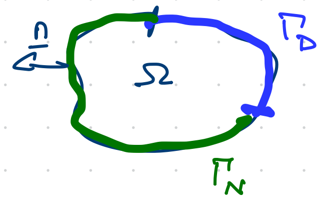
$m=0,1$ $\qquad\qquad$ $1 < s \leq r+1$

$r =$ degree of approximation.

(Lax-Milgram, quasi-optimality, interpolation estimates, duality argument)

2.4 Dealing with generic boundary conditions.

$$\begin{cases} -\operatorname{div}(A\nabla u) = f \\ \gamma_{\Gamma_D}(u) = g_D \quad \text{on } \Gamma_D \\ (A\cdot\nabla u)\cdot n = g_N \quad \text{on } \Gamma_N \end{cases}$$



At the continuous level $\quad R : \underbrace{H^{1/2}(\Gamma_D)}_{\text{the space of traces of } H^1} \to H^1(\Omega)$

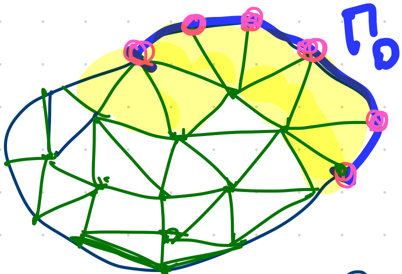
$R(g_D)$ continuous extension of the boundary data.

$u = u_0 + R(g_D) \qquad u_0 \in H^1_0(\Omega)$

Find $u_0 \in H^1_{0,\Gamma_D}(\Omega)$: $\quad \forall v \in H^1_{0,\Gamma_D}(\Omega)$

$$\int_\Omega A\nabla(u_0 + R(g_D))\nabla v = \int_\Omega f v + \int_{\Gamma_N} g_N v$$

$$\boxed{\int_\Omega A\nabla u_0 \nabla v = \int_\Omega f v + \int_{\Gamma_N} g_N v - \int_\Omega A\nabla R(g_D)\nabla v}$$



$\mathbb{P}_1$ finite elements

v_i , $i \in \mathcal{I}$

vertices of $\mathcal{T}_h$.

$$\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_D$$

$\mathcal{I}_D$ = indices of the vertices lying on Γ_D

$$u_h = \underbrace{\sum_{i \in \mathcal{I}_0} u_i \varphi_i}_{\in H^1_{0,\Gamma_D}(\Omega)} + \underbrace{\sum_{e \in \mathcal{I}_D} g(v_e)\varphi_e}_{R_h(I_h g_D)}$$

Find $u_{0,h} \in V_h \cap H^1_{0,\Gamma_D}$ $\quad \forall v_h \in V_h \cap H^1_{0,\Gamma_D}$

$$\boxed{\int_\Omega A \nabla u_{0,h} \cdot \nabla v_h = \int_\Omega f v_h + \int_{\Gamma_N} g_N v_h + - \int_\Omega A \nabla R_h(I_h g_D) \nabla v_h.}$$

Error estimates

$a(u,v) = \int A \nabla u \nabla v$

$u = u_0 + R(g_D)$

$u_h = u_{0,h} + R_h(I_h g_D)$

$$\begin{cases} a(u, v) = \int_\Omega f v + \int_{\Gamma_N} g_N v \\ a(u_h, v_h) = \quad " \quad\quad " \end{cases}$$

$u - u_h \notin H^1_{0,\Gamma_D}(\Omega)$ $\Rightarrow$ $v = u - u_h$ is not a test funct

$$a(u-u_h, v_h) = 0 \qquad \forall v_h \in V_h \cap H^1_{0,\Gamma_D}(\Omega)$$

$$0 = a(u-u_h, u_h-w_h) = a(u-w_h, u_h-w_h) - a(u_h-w_h, u_h-w_h)$$

$$u_h \in \{ v_h \in V_h : u_h|_{\Gamma_D} = I_h g_D \} = V_{h,g}$$

By selecting $w_h \in V_{h,g}$. $\Rightarrow u_h - w_h|_{\Gamma_D} = 0$.

$$\alpha \|u_h-w_h\|^2_{H^1} \leq a(u_h-w_h, u_h-w_h) = $$
$$= a(u-w_h, u_h-w_h)$$

$$\Rightarrow \quad \|u_h-w_h\|_{H^1} \leq \frac{M}{\alpha} \|u-w_h\|_{H^1} \quad \forall w_h$$

$$\|u-u_h\|_{H^1} \leq \|u-w_h\|_{H^1} + \|u_h-w_h\|_{H^1}$$
$$\leq \|u-w_h\|_{H^1} + M/\alpha \|u-w_h\|_{H^1}$$
$$\leq \left(1+\frac{M}{\alpha}\right) \inf_{w_h \in V_{h,g}} \|u-w_h\|_{H^1}$$

$$\underbrace{w_h = I_h u}_{} \qquad w_h|_{\Gamma_D} = I_h u|_{\Gamma_D} = I_h g_D$$

this is an admissible choice, because $w_h \in V_{h,g}$!!

$$\|u-u_h\|_{H^1} \leq \left(1+\frac{M}{\alpha}\right) \|u-I_h u\|_{H^1}$$
$$\left(1+M/\alpha\right) \quad h^{s-1} \quad |u|_{H^s}$$
$$1 < s \leq r+1$$

Conclusion : always take the interpolant of the Dirichlet boundary conditions for the discrete problem.

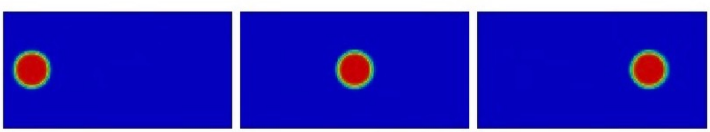
CHAPTER 3 Convection diffusion equations.

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u) + \underline{\beta} \cdot \nabla u = f \\ \gamma_{\partial\Omega}(u) = g. \end{cases}$$

$\underline{\beta} = (\beta_1, \beta_2)$ field transport.

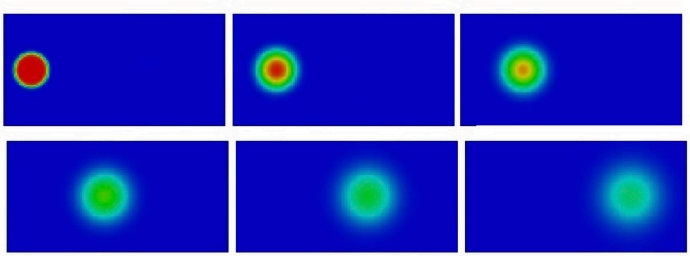
$\operatorname{div}(\underline{\beta}) = 0$ $\qquad \dfrac{\partial}{\partial x_1}\beta_1 + \dfrac{\partial}{\partial x_2}\beta_2 = 0.$

$|\underline{\beta}| \gg \varepsilon$ $\qquad$ Transport dominated problem
convection dominated problem.



$$\frac{\partial u}{\partial t} + \underline{\beta} \cdot \nabla u = 0$$

transport problem



$$\cancel{\frac{\partial u}{\partial t}} - \varepsilon \Delta u + \underline{\beta} \cdot \nabla u = 0$$

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u) + \underline{\beta} \cdot \nabla u = f & \Omega \\ \gamma(u) = 0. & \partial\Omega \end{cases} \qquad |\underline{\beta}| \gg \varepsilon$$

Weak formulation. $u \in H_0^1(\Omega)$ $\forall v \in H_0^1(\Omega)$

$$\underbrace{\int_\Omega \varepsilon \nabla u \cdot \nabla v + \int_\Omega \underline{\beta} \cdot \nabla u \, v}_{a(u,v)} = \int f v$$

$$a(u,u) \geq \varepsilon \, \|u\|_{H^1}^2$$

$$a(u,v) \leq (\varepsilon + |\underline{\beta}|) \|u\|_{H^1} \cdot \|v\|_{H^1} \leq C \cdot |\underline{\beta}| \, \|u\|_{H^1} \|v\|_{H^1}$$

$$\frac{|\underline{\beta}|}{\varepsilon} = \text{Péclet number.} = Pe \qquad M/\alpha$$

Standard Galerkin error estimates.

$$\|u - u_h\| \leq Pe \cdot \|u - I_h u\|$$

3.1 1D convection diffusion problems: finite difference method.

$$\begin{cases} -(\varepsilon u')' + \beta u' = f \\ u(a) = g_a \quad u(b) = g_b \end{cases}$$

$$u'' \Rightarrow \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \qquad u' \Rightarrow \frac{u_{j+1} - u_{j-1}}{2h}$$

$$\boxed{-\varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \beta \frac{u_{j+1} - u_{j-1}}{2h} = f_j}$$

$AU = \underline{f}$ $\qquad A = \text{tri-diag}\left(-\frac{\varepsilon}{h} - \frac{\beta}{2}, \frac{2\varepsilon}{h}, -\frac{\varepsilon}{h} + \frac{\beta}{2}\right)$

(after multiplication by h)

$\underline{f} = h f_j$

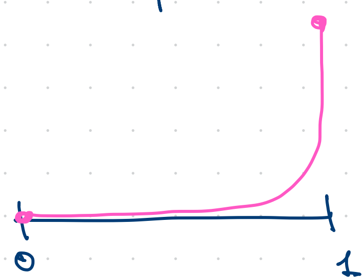
$\rightsquigarrow$ + boundary conditions.

Let's look at $f = 0$, $g_a = 0$ $\quad g_b = 1$.

$a = 0,\ b = 1$ $\qquad \beta > 0$

$$\begin{cases} -\varepsilon u'' + \beta u' = 0. \\ u(0) = 0 \quad u(1) = 1 \end{cases}$$

$$u(x) = \frac{e^{\beta x/\varepsilon} - 1}{e^{\beta/\varepsilon} - 1}$$



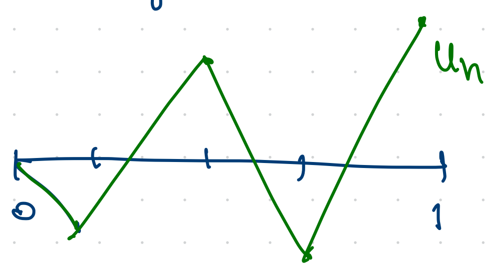
Finite difference method...

$$u_j = \frac{\Gamma^j - 1}{\Gamma^N - 1} \ ; \quad \Gamma = \frac{1 + Pe_h}{1 - Pe_h}$$

$Pe_h = \frac{\beta h}{2\varepsilon}$ discrete Péclet number.

$Pe_h > 1$ $\qquad \Gamma$ is negative !

$\to$ u_j is oscillating on the mesh.



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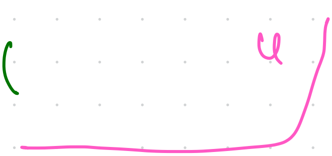
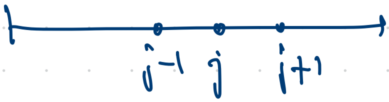


if $\quad Pe_h < 1$ the discrete solution is monotone and a good approximation of u

$Peh = \frac{\beta}{2\varepsilon} \cdot h$ $\quad Peh < 1$ is a condition on h!

Upwind fix

$$- \varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \beta \cancel{\frac{u_{j+1} - u_{j-1}}{2h}}$$



upwind first derivative $\frac{u_j - u_{j-1}}{h}$

$$u_j = \frac{(1 + Peh)^j - 1}{(1 + Peh)^N - 1}$$

→ • monotone solution

• stable solution

• an approximation of u.

$$\frac{u_j - u_{j-1}}{h} - \frac{u_{j+1} - u_{j-1}}{2h} = \underbrace{-\frac{h}{2} \left[\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \right]}_{\text{diffusion!}}$$

upwind scheme re-writes

$$\left(-\varepsilon - \beta \frac{h}{2} \right) \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \beta \frac{u_{j+1} - u_{j-1}}{2h}$$

$$\widetilde{Peh} = \frac{\beta}{\varepsilon + \frac{\beta h}{2}} \cdot \frac{h}{2} = \frac{\beta h}{2\varepsilon + \beta h} < 1 \text{ !}$$

3.2 Finite element discretisations.

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u) + \underline{\beta} \cdot \nabla u = f \\ \gamma(u) = g \end{cases}$$

$$\|u - u_h\|_{H^1} \leq Pe \cdot \|u - I_h u\|_{H^1} \lesssim Pe \cdot h$$

$$Peh = \frac{\beta}{2\varepsilon} \cdot h \qquad r = 1 \qquad \|u - I_h u\|_1 \leq Ch$$

If $Peh > 1$ the error is bounded by a constant bigger than 1 !

Artificial viscosity.

$$a_h(u_h, v_h) = \int_\Omega \left(\varepsilon + \gamma \frac{|\beta| h}{2} \right) \nabla u \cdot \nabla v + \int_\Omega \underline{\beta} \cdot \nabla u \, v$$

γ = a parameter I can try to tune.

Stream line diffusion

$$a_h(u_h, v_h) = \int_\Omega \varepsilon \nabla u_h \nabla v_h + \underline{\beta} \nabla u_h v_h + \gamma \frac{h}{2|\underline{\beta}|} \int (\beta \cdot \nabla u_h) \cdot (\underline{\beta} \cdot \nabla v_h)$$

if u is the solution of the continuous problem...

$$a_h(u, v_h) \neq \int f v_h$$

All these methods are <u>not consistent</u> !