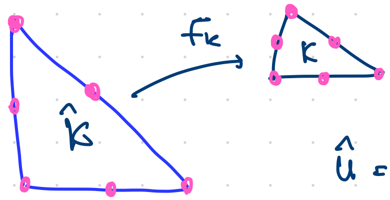

$\hat u = u \circ F_K$

$\hat I(\hat u) \qquad I_K u = \hat I(u \circ F_K)$

$\hat I(\hat p) = \hat p \qquad \hat p \in \mathbb{P}^r(\hat K)$

interpolation estimate for $m = 1$ $\qquad$ K triangle of $\mathcal{T}_h$

- Scaling

$$|u|_{H^1(K)} \le C\, h_K^{-1} h_K^{1} |\hat u|_{H^1(\hat K)}$$

$$|\hat u|_{H^{r+1}(\hat K)} \le C\, h_K^{r+1} h_K^{-1} |u|_{H^{r+1}(K)}$$

- Bramble–Hilbert

$$\hat \lambda = 1 - \hat I$$

$$\hat u - \hat I \hat u = \hat u + \hat p - \hat I(\hat u + \hat p)$$

$$\hat\lambda(\hat p) = 0 \quad \forall \hat p \in \mathbb{P}^r(\hat K)$$

$$|\hat u - \hat I \hat u|_{H^1} \le C \inf_{\hat p \in \mathbb{P}^r(\hat K)} \|\hat u + \hat p\|_{H^{r+1}(\hat K)}$$

- Deny–Lions (generalized Poincaré inequality)

$$\inf_{\hat p \in \mathbb{P}^r(\hat K)} \|\hat u + \hat p\|_{H^{r+1}(\hat K)} \le C\, |\hat u|_{H^{r+1}(\hat K)}$$

$$|u - I_K u|_{H^1(K)} \underset{\substack{\uparrow \\ \text{scaling } H^1}}{\le} C\, |\hat u - \hat I \hat u|_{H^1(\hat K)}$$

$$\underset{\substack{\uparrow \\ \text{Bramble Hilbert}}}{\le} C \inf_{\hat p \in \mathbb{P}^r(\hat K)} \|\hat u + \hat p\|_{H^{r+1}(\hat K)}$$

$$\underset{\text{Deny Lions}}{\leq} \quad C \quad |\hat{u}|_{H^{r+1}(\hat{K})}$$

$$\leq \quad C \, h_K^{r} \quad |u|_{H^{r+1}(K)}$$

the final estimate

$$|u - I_K(u)|_{H^1(K)} \leq C \, h_K^{r} \, |u|_{H^{r+1}(K)}$$

<u>local</u> interpolation estimate in H^1

$m=0$

$$\| u - I_K(u) \|_{L^2(K)} \leq C \, h_K^{r+1} \, |u|_{H^{r+1}(K)}$$

local i.e. in L^2

$$\sum_{K \in \mathcal{T}_h} \left(|u - I_K(u)|^2_{H^1(K)} \right) = |u - I(u)|^2_{H^1(\Omega)}$$

$$I(u)|_K = I_K(u)$$

$$\sum_{K \in \mathcal{T}_h} \left\{ |u - I_K(u)|^2_{H^1(K)} \leq C \, h_K^{2r} \, |u|^2_{H^{r+1}(K)} \right\}$$

$$|u - I(u)|^2_{H^1(\Omega)} \leq C \, h^{2r} \, |u|^2_{H^{r+1}(\Omega)}$$

$$\boxed{|u - I(u)|_{H^1(\Omega)} \leq C h^r |u|_{H^{r+1}(\Omega)}}$$

$$\boxed{\| u - I(u) \|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}}$$

$$\boxed{|u - I(u)|_{H^m(\Omega)} \leq C h^{r+1-m} |u|_{H^{r+1}(\Omega)} \quad m = 0, 1 \ldots r}$$

global interpolation estimate

$r = 1$ $\quad |u - I^1(u)|_{H^m(\Omega)} \leq C h^{2-m} |u|_{H^2(\Omega)}$

$m = 0, 1$

$r = 2$ $\quad |u - I^2(u)|_{H^m(\Omega)} \leq C h^{3-m} |u|_{H^3(\Omega)}$

$m = 0, 1$ (m = 2 not used)

If the function u is regular (say H^3) then it is better to use higher polynomials degrees ($r = 2$ if $u \in H^3$, $r = 3$ if $u \in H^4$)

if u is not regular, the estimates as is do not work...

Indeed, by sophisticated math tools:

if $u \in H^s(\Omega)$ $\quad 2 \leq s \leq r+1$ then

$$|u - I^r(u)|_{H^m(\Omega)} \leq C\, h^{s-m} \, |u|_{H^s(\Omega)}$$

$I^r(u)$ interpolation of u on p.w. polynomials of degree r

On the Galerkin problem $\quad \|u-u_h\|_{H^1(\Omega)} \leq \dots$

$$\|u-u_h\|_{H^1(\Omega)} \leq M/\alpha \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}$$

$u \in H^s(\Omega) \quad s \geq 2 \qquad v_h = I^r(u)$

$$\boxed{\|u - u_h\|_{H^1(\Omega)}} \leq \frac{M}{\alpha} \|u - I^r(u)\|_{H^1(\Omega)}$$

$$\leq C \frac{M}{\alpha} \cdot h^{s-1} \, |u|_{H^s(\Omega)}$$

$u \in H^{r+1}$

$$\boxed{\leq C \frac{M}{\alpha} h^r \, |u|_{H^{r+1}(\Omega)}}$$

$\|u-u_h\|_{L^2(\Omega)}$? what do I know ?

$$\|u - u_h\|_{L^\infty(\Omega)} := \max_{x \in \Omega} |u(x) - u_h(x)| \quad ?$$

About the $\|u-u_h\|_{L^2(\Omega)}$

Duality argument - Aubin-Nitsche trick

Our problem $\quad u \in V \quad a(u,v) = \int f v \quad \forall v \in V$

adjoint problem $\quad \varphi \in V \quad a(v,\varphi) = \int f v \quad \forall v \in V$

(if $a(\cdot,\cdot)$ is symmetric, then the adjoint problem is the same as the original one).

adjoint, $f = u - u_h \qquad a(v,\varphi) = \int (u-u_h)\cdot v \quad \forall v \in V$

φ is the solution of the adj. problem with $u-u_h$ as rhs

$v = u - u_h \qquad a(u-u_h,\varphi) = \int (u-u_h)^2 = \|u-u_h\|^2_{L^2(\Omega)}$

$$\|u-u_h\|^{\cancel{2}}_{L^2(\Omega)} = a(u-u_h,\varphi) \underset{\substack{\uparrow \\ \text{Galerkin} \\ \text{orthogonality}}}{=} a(u-u_h,\varphi - w_h) \quad \forall w_h \in V_h$$

$$\leq M \|u-u_h\|_{H^1} \cdot \|\varphi - w_h\|_{H^1(\Omega)} \qquad (\bullet)$$

$\varphi \in H^2(\Omega)$ and moreover $\qquad \|\varphi\|_{H^2(\Omega)} \leq C \|u-u_h\|_{L^2(\Omega)}$

$w_h = I^r(\varphi) \qquad$ (continuous dependence on the data)

$$(\bullet) \quad \leq M \|u-u_h\|_{H^1} \cdot \|\varphi - I^r(\varphi)\|_{H^1}$$

$$\leq M \|u-u_h\|_{H^1} \cdot h \cdot |\varphi|_{H^2(\Omega)}$$

$$\leq M \cdot \|u-u_h\|_{H^1} \cdot h \cdot \cancel{\|u-u_h\|}_{L^2(\Omega)}$$

$$\boxed{\|u-u_h\|_{L^2(\Omega)} \leq \cdot\, Mh\, \|u-u_h\|_{H^1}}$$

L^2 error estimate