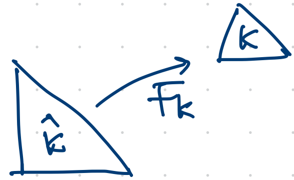
Structure of the stiffness matrix

$F_K \underline{\hat{x}} = B_K \underline{\hat{x}} + \underline{b}$



$v_h\big|_K$ finite element function

$\hat{v}(\hat{x}) = v_h\big|_K (F_K \underline{\hat{x}})$

$\hat{v}(\hat{x}) = \sum_{\ell=1}^{\hat{N}_e} \hat{v}_\ell \underbrace{\hat{\varphi}_e}_{\text{basis function}}$

$\| \hat{v} \|_{L^2(\hat{K})} \simeq |\underline{\hat{v}}|_{\ell_2}$

$\exists C_1, C_2 :$
$C_1 \|\hat{v}\|_{L^2(\hat{K})} \leq |\underline{\hat{v}}|_{\ell_2} \leq C_2 \|\hat{v}\|_{L^2(\hat{K})}$

on the element K

$v_h\big|_K = \sum v_\ell \varphi_e \qquad v_h\big|_K (F_K \hat{x}) = \sum v_e \hat{\varphi}_e = \hat{v}$

$\int_K |v_e|^2 = \int_{\hat{K}} \overbrace{|v_e(F_K \hat{x})|^2}^{\hat{v}} \det B_K$

$= \underbrace{|\det B_K|}_{h_K^2} \|\hat{v}\|^2_{L^2(\hat{K})}$

depends on the dimension d, $d = 2$

$$\boxed{\|v_e\|_{L^2(K)} \simeq h_K \|\hat{v}\|_{L^2(\hat{K})} \simeq h_K |\underline{v}|_{\ell_2}}$$

$\int_K |\nabla v_h|^2 = \int_{\hat{K}} |B_K^{-T} \nabla \hat{v}|^2 \det B_K$

$\leq \underbrace{|\det B_K|}_{h_K^d = h_K^2} \underbrace{\|B_K^{-T}\|^2}_{(1/\rho_K)^2} \cdot \int_{\hat{K}} |\nabla \hat{v}|^2$

$$\|\nabla v_h\|^2_{L^2(K)} \le h_K^2 \cdot \frac{1}{\rho_K^2} \cdot \|\nabla \hat{v}\|^2_{L^2(\hat{K})}$$

$$\le C\, h_K^2 \; \frac{1}{\rho_K^2} \; \|\hat{v}\|^2_{L^2(\hat{K})}$$

equivalence of norms in finite dimension

$$\le C\, \cancel{h_K^2} \; \frac{1}{\rho_K^2} \; \cancel{h_K^{-2}} \; \|v_h\|^2_{L^2(K)}$$

$$\boxed{\|\nabla v_h\|_{L^2(K)} \le C \, \frac{1}{\rho_K} \, \|v_h\|_{L^2(K)}}$$

Local inverse inequality

$$\frac{1}{\rho_K} \simeq h_K^{-1}$$

Sum this inequality over all triangles.

$$\sum_K \|\nabla v_h\|^2_{L^2(K)} \le C \sum_K h_K^{-2} \, \|v_h\|^2_{L^2(K)}$$

$$\|\nabla v\|^2_{L^2(\Omega)} \le C \underbrace{\left(\max_K h_K^{-2}\right)}_{\left(\min_K h_K\right)^{-2}} \|v_h\|^2_{L^2(\Omega)}$$

If $\min_K h_K \ge C \max_K h_K$

$$\boxed{\|\nabla v_h\|_{L^2(\Omega)} \le C' \, h^{-1} \, \|v_h\|_{L^2(\Omega)}}$$

$\forall v_h \in V_h$.

inverse inequality.

<u>Remark</u> $u \in H^1(\Omega)$? $\cancel{\|\nabla u\|_0 \le C \|u\|_0}$ always wrong!

instead if $u \in H^1_0(\Omega)$ $\|u\|_0 \underset{\text{Poincaré}}{\le} C_p \|\nabla v\|_0$

Going back to $\underline{\underline{A}}$ $\Rightarrow$ $\underline{\underline{A}}_{ij} = \int_\Omega A \nabla \varphi_i \nabla \varphi_j$

1) $\underline{\underline{A}}$ is symmetric and positive definite.

proof I want to prove: ${}^T\underline{u} \, \underline{\underline{A}} \, \underline{u} \ge \alpha_A |\underline{u}|^2_{\ell_2}$

$${}^T\underline{u} \, \underline{\underline{A}} \, \underline{u} = a(u,u) \underset{\text{coercivity.}}{\ge} \alpha \|u\|^2_{H^1(\Omega)} \ge \alpha \|u\|^2_{L^2(\Omega)}$$

$$\ge \underbrace{\alpha \, h^2}_{\alpha_A} \cdot |\underline{u}|^2_{\ell_2}$$

the inequality above, assumed over all the elements.

$$u = \sum u_i \varphi_i$$
$$v = \sum v_j \varphi_j$$
$$\{u_i\}_i = \underline{u}, \quad \{v_j\}_j = \underline{v}$$

2) $\mathrm{cond}(\underline{\underline{A}}) \le C \left(1 + 1/h^2\right)$

$${}^T\underline{v} \, \underline{\underline{A}} \, \underline{u} = a(u,v) \le M \|u\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}$$

$u = v$ $\quad {}^T\underline{u} \, \underline{\underline{A}} \, \underline{u} \le M \|u\|^2_{H^1}$

$$\le M \left(\|u\|^2_{L^2} + \frac{1}{h^2} \|u\|^2_{L^2(\Omega)} \right)$$

inverse inequality.

$$\le M \left(1 + 1/h^2\right) \cdot \|u\|^2_{L^2(\Omega)}$$

$$\le M \left(1 + 1/h^2\right) \cdot h^2 \cdot |\underline{u}|^2_{\ell_2}$$

$$\alpha\, h^2 \; |\underline{u}|^2_{\ell_2} \le {}^T\underline{u}\,\underline{\underline{A}}\,\underline{u} \le M\left(1+\frac{1}{h^2}\right)\cdot h^2\; |\underline{u}|^2_{\ell_2}$$

$$\mathrm{cond}(A) \simeq \frac{M}{\alpha}\left(1+\frac{1}{h^2}\right)$$

3) $\underline{\underline{A}}$ is sparse. – basis functions are local !!

2.3 Error estimate

Galerkin method + Lax – Milgram.

$$\| u - u_h \|_1 \le \frac{M}{\alpha} \inf_{v_h \in V_h} \| u - v_h \|_1$$

$V_h = \{ u_h \in H^1(\Omega) \quad u_h|_K \in \mathbb{P}^r(K) \;$ + boundary conditions$\}$.

V_h : $\dim(V_h) = N_h$

$\varphi_1 \dots \varphi_{N_h}$ basis functions.

$\xi_1 \dots \xi_{N_h}$ vertices or mid point of edges ..... (evaluation points)

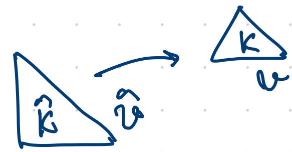
$$\underbrace{I_h u}_{} = \sum_{i=1}^{N_h} u(\xi_i)\,\varphi_i$$

interpolation operator, it can be applied to all functions than can be evaluated on points.

$$\|u-u_h\|_1 \le \frac{M}{\alpha} \inf_{v_h} \|u-v_h\|_1 \le \frac{M}{\alpha} \|u - I_h u\|_1$$

$\|u - I_h u\|_1$ = interpolation error.

- **Scaling** $\quad v(F_k \hat{x}) = \hat{v}(\hat{x})$



$$\|v\|_{L^2(K)} \simeq h_K^2 \|\hat{v}\|_{L^2(\hat{K})}$$

$$\|\nabla v\|_{L^2(K)} \simeq \|\nabla \hat{v}\|_{L^2(\hat{K})}$$

(exercise, starting from what we said before)

$$|u|^2_{H^m(K)} = \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(K)}$$

$$|\hat{u}|_{H^m(\hat{K})} = \sum_{|\alpha|=m} \|\hat{D}^\alpha \hat{u}\|^2_{L^2(K)}$$

Scaling :

$$|v|_{H^m(K)} \le C \|B_k^{-1}\|^m (\det B_k)^{1/2} |\hat{v}|_{H^m(\hat{K})}$$

$$|\hat{v}|_{H^m(\hat{K})} \le C \|B_k\|^m (\det B_k)^{-1/2} |v|_{H^m(K)}$$

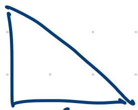
and with respect to the mesh size (for regular meshes)

$$|v|_{H^m(K)} \le C h_K^{-m} h_K |\hat{v}|_{H^m(\hat{K})}$$

dimension dependent (it is $h_K^{d/2}$)

$$|\hat{v}|_{H^m(\hat{K})} \le C h_K^m h_K^{-1} \cdot |v|_{H^m(K)}$$

• Bramble-Hilbert Lemma



$\hat K$

$\hat\varphi_1 \dots \hat\varphi_{\hat N}$

$\hat\xi_1 \dots \hat\xi_{\hat N}$

$$\hat I \hat u = \sum \hat u(\hat\xi_i)\hat\varphi_i$$

$$\underbrace{(I-\hat I)}_{\text{operator}} u = 0 \qquad \forall u \in \mathbb{P}^r(\hat K)$$

operator : $C^0(\hat K) \to \mathbb{P}^r(\hat K)$

$$\boxed{\begin{array}{l} \text{Given } \hat\lambda \text{ linear operator:} \\ \hat\lambda : H^s(\hat K) \to H^m(\hat K) \qquad \hat\lambda(\hat p) = 0 \quad \forall \hat p \in \mathbb{P}^r(\hat K) \\ \text{Then, it holds that:} \\ |\hat\lambda(\hat v)|_{H^m(\hat K)} \le C \inf_{\hat p \in \mathbb{P}^r(\hat K)} \|\hat v + \hat p\|_{H^s(\hat K)} \end{array}}$$

<u>proof</u> $\hat v \in H^s(\hat K) = \{\hat u \in L^2 : |\hat u|_{H^s(\hat K)} < \infty\}$.

$$\hat\lambda(\hat u) = \hat\lambda(\hat u + \hat p) \qquad \forall \hat p$$

linearity of the operator

$$|\hat\lambda(\hat u)|_{H^m(\hat K)} = |\hat\lambda(\hat u + \hat p)|_{H^m(\hat K)}$$

$$\le C \, \|\hat u + \hat p\|_{H^s(\hat K)} \qquad \forall \hat p \in \mathbb{P}^r(\hat K)$$

$$\leq C \inf_{\hat{p} \in \mathbb{P}^r(\hat{K})} \|\hat{u} + \hat{p}\|_{H^s(\hat{K})}$$

- Deny - Lions Lemma $\forall r \geq 0$.

$$\inf_{\hat{p} \in \mathbb{P}^r(\hat{K})} \|\hat{v} + \hat{p}\|_{H^{r+1}(\hat{K})} \leq C |\hat{v}|_{H^{r+1}(\hat{K})}$$

$H^{r+1}(\hat{K}) = \{ \hat{u} \in L^2(\Omega) :$ all derivatives of order up to $r+1$ are in $L^2 \}$.

We do not prove this one, but we make a remark:

$r = 0$

$$\inf_{\hat{p} \in \mathbb{R}} \|\hat{v} + \hat{p}\|^2_{H^1} = \inf_{\hat{p} \in \mathbb{R}} \left(\|\hat{v} + \hat{p}\|^2_{L^2} + |\nabla \hat{v}|^2_{L^2} \right)$$

$$= \left(\inf_{\hat{p} \in \mathbb{R}} \|\hat{v} + \hat{p}\|_{L^2} \right) + |\nabla \hat{v}|_{L^2}$$

$$\leq C |\nabla \hat{v}|_{L^2(\hat{K})}$$

... this is just a Poincaré estimate then!

$\hat{p} = -\fint_K \hat{v}$

$$\inf_{\hat{p}} \|\hat{v} + \hat{p}\|_{L^2} = \|\hat{v} - \fint_K \hat{v}\|_{L^2(\hat{K})}$$

$$\leq C_P |\nabla \hat{v}|_{L^2(\hat{K})}$$

$\underline{r=1}$ $$\inf_{p_1 \in \mathbb{P}^1(\hat K)} \|\hat v + p_1\|_{H^2(\hat K)} \le C \, |\hat v|_{H^2(\hat K)}$$

$$\|\hat v + p_1\|^2_{H^2(\hat K)} = \underbrace{\|\hat v + p_1\|^2_{H^1(\hat K)}}_{\le C\, |\hat v|_{H^2(\hat K)}} + |\hat v|_{H^2(\hat K)}$$

→ Deny Lions is a generalisation of the Poincaré inequality to higher order!

Now, we go to the interpolation error

$\hat K$ $\quad \hat\varphi_1 \dots \hat\varphi_N$, $\quad \hat\xi_1 \dots \hat\xi_N$

K $\quad \varphi_1 \dots \varphi_N$, $\quad \xi_1 \dots \xi_N$

$$\hat I \hat u = \sum_{i=1}^{N} \hat u(\hat\xi_i)\, \hat\varphi_i$$

$$I_K u = \sum_{i=1}^{N} u(\xi_i)\, \varphi_i$$

$$\boxed{|u - I_K u|_{H^m(K)}} \underset{\text{scaling}}{\le} C\, h_K^{-m} \cdot h_K\, |\hat u - \hat I \hat u|_{H^m(\hat K)}$$

$m=0$, $m=1$

$$\underset{\text{Bramble Hilbert}}{\le} C\, h_K^{1-m} \inf_{\hat p \in \mathbb{P}^r(\hat K)} \|\hat u - \cancel{\hat I \hat u} + \hat p\|_{H^{r+1}(\hat K)}$$

$$\underset{\text{Deny Lions}}{\le} \boxed{C \, h_K^{1-m} \, \underbrace{|\hat{u}|_{H^{r+1}(\hat{K})}}}$$

$$\underset{\text{Scaling}}{\le} C \, h_K^{1-m} \, \underbrace{h_K^{-1} \, h_K^{r+1} \, |u|_{H^{r+1}(K)}}$$

$$\le C \, h_K^{r+1-m} \, |u|_{H^{r+1}(K)}$$