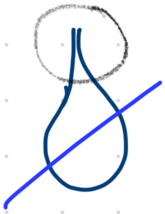
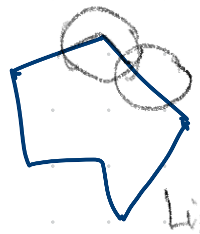
1.4 Wellposedness in $d = 2, 3 \ldots n$

$\Omega \subseteq \mathbb{R}^d$ bounded, open, Lipschitz



not Lipschitz



Lipschitz

$L^2(\Omega)$ $\quad L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R}$ "essentially" bounded$\}$.

$H^1(\Omega) = \{ u \in L^2(\Omega) \;:\; \nabla u \in L^2(\Omega)^d \}$.

$L^2(\partial\Omega) = \{ v : \partial\Omega \to \mathbb{R} \;:\; \int_{\partial\Omega} |v|^2 < +\infty \}$.

$H^1(\Omega) \not\subseteq C^0(\Omega)$!! $\quad d = 2, 3 \ldots$

<u>trace theorem</u>

$$\gamma_{\partial\Omega} : H^1(\Omega) \to L^2(\partial\Omega)$$
$$u \mapsto u|_{\partial\Omega}$$

$$\| \gamma_{\partial\Omega}(u) \|_{L^2(\partial\Omega)} \leq C \; \| u \|_{H^1(\Omega)}$$

continuity of trace

<u>remark</u>

$$\gamma_{\partial\Omega}(H^1(\Omega)) \subsetneq L^2(\partial\Omega)$$

<u>definition</u>

$$\gamma_{\partial\Omega}(H^1(\Omega)) = H^{1/2}(\partial\Omega)$$

$$\| g \|_{1/2, \partial\Omega} = \inf_{\substack{u \in H^1(\Omega) \\ \gamma_{\partial\Omega}(u) = g}} \| u \|_{H^1(\Omega)}$$

remark $\quad u \in H^1(\Omega) \quad \Gamma \subseteq \partial\Omega \quad \gamma_\Gamma(u) \in H^{1/2}(\Gamma)$

definition $\boxed{H_0^1(\Omega)} = \{u \in H^1(\Omega) \; : \; \gamma_{\partial\Omega}(u) = 0\}.$

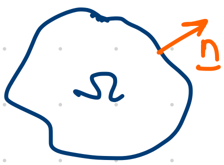
$\Gamma \subseteq \partial\Omega$ $\boxed{H_{0,\Gamma}^1(\Omega)} = \{u \in H^1(\Omega) \; : \; \gamma_\Gamma(u) = 0\}.$

$$\boxed{\begin{cases} -\operatorname{div}(\mathbb{K}\nabla u) = f \\ \gamma_{\partial\Omega}(u) = 0 \end{cases}} \quad (1)$$

diffusion equation.
Dirichlet boundary conditions.

$\mathbb{K} = \mathbb{K}(\underline{x}) \in L^\infty(\Omega)^{d\times d}$
symmetric positive def.
$\underline{a}^T \mathbb{K} \underline{a} \geq \beta > 0 \quad \forall \underline{x} \in \Omega.$
$\forall \underline{a} \in \mathbb{R}^d$

$$\operatorname{div}(\underline{a}) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \dots + \frac{\partial a_n}{\partial x_n}$$



We seek for weak solutions of (1)

$u \in H_0^1(\Omega)$ " $\displaystyle\int_\Omega$ " $-\operatorname{div}(\mathbb{K}\nabla u)\,\varphi = \displaystyle\int_\Omega f\varphi \quad \forall \varphi \in H_0^1(\Omega)$

integration by parts

$$\int_\Omega -\operatorname{div}(\mathbb{K}\nabla u)\,\varphi = \int_\Omega \mathbb{K}\nabla u \cdot \nabla\varphi - \underbrace{\cancel{\int_{\partial\Omega} (\mathbb{K}\nabla u)\cdot\underline{n}\,\varphi}}_{=0.}$$

Variational formulation: $f \in L^2(\Omega)$

$$\boxed{\begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \mathbb{K}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in H_0^1(\Omega) \end{array}} \quad (2)$$

<u>theorem</u> : Problem (2) is well-posed.

Lax-Milgram lemma had ~~two~~ three main assumptions: $a(u,v)=\int_\Omega fv$

1) $f \in L^2(\Omega)$ ✓

2) $a(u,v) = \int_\Omega \mathbb{K} \nabla u \cdot \nabla v$

- $|a(u,v)| \leq M \|u\|_1 \cdot \|v\|_1$

3) $a(u,u) = \int_\Omega \mathbb{K} |\nabla u|^2$

- $a(u,u) \geq \alpha \|u\|_1^2$

2) $$\int_\Omega \mathbb{K} \nabla u \cdot \nabla \varphi \leq \|\mathbb{K}\|_\infty \cdot \|\nabla u\|_0 \cdot \|\nabla \varphi\|_0$$
$$\leq \underbrace{\|\mathbb{K}\|_\infty}_{M} \cdot \|u\|_1 \cdot \|\varphi\|_1$$ ✓

3) $$\int_\Omega \mathbb{K} |\nabla u|^2 \geq \underbrace{\beta}_{\text{smallest eigenvalue of } \mathbb{K}} \|\nabla u\|_0^2 \quad \cdots \quad \geq \alpha \|u\|_1^2$$

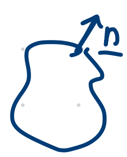
$\alpha = \alpha(\beta, C_p)$

Poincaré inequality $\forall u \in H^1_0(\Omega)$ it holds ✓
$$\|u\|_0 \leq C_p \|\nabla u\|_0$$

$\rightarrow$ the problem (2) is well posed

Other boundary conditions

- $\gamma_{\partial\Omega}(u) = 0$ homogeneous Dirichlet boundary condition.
- $\gamma_{\partial\Omega}(u) = g_D$ non-hom. Dirichlet. $g_D \in H^{1/2}(\partial\Omega)$
- $"(\mathbb{K} \nabla u) \cdot \underline{n}\big|_{\partial\Omega}" = \gamma_N(u)$ Neumann boundary trace



$\underline{n}$ = normal vector

$\gamma_N(u) = g_N \qquad g_N \in L^2(\partial\Omega)$

- mixed boundary conditions.

- Non-homogeneous Dirichlet boundary conditions

$$(3)\ \begin{cases} -\operatorname{div}(\mathbb{K}\nabla u) = f & f \in L^2(\Omega) \\ \gamma_{\partial\Omega}(u) = g_D & g_D \in H^{1/2}(\partial\Omega) \end{cases}$$

$$\exists\ R(g_D) \in H^1(\Omega) \quad : \quad \gamma_{\partial\Omega}(R(g_D)) = g_D$$

$$\|R(g_D)\|_1 \leq C\ \|g_D\|_{1/2}$$

u of (3) $\qquad u = u_0 + R(g_D) \qquad u_0 \in H^1_0(\Omega)$

$$\gamma_{\partial\Omega}(u) = \gamma_{\partial\Omega}(u_0 + R(g_D)) = 0 + \gamma_{\partial\Omega}(R(g_D)) = g_D$$

Find $u = u_0 + R(g_D)$ such that

$$(4)\quad \int_\Omega \mathbb{K}\nabla(u_0 + R(g_D)) \cdot \nabla\varphi = \int_\Omega f \cdot \varphi \quad \forall \varphi \in H^1_0(\Omega)$$

$$\boxed{\int_\Omega \mathbb{K}\nabla u_0 \cdot \nabla\varphi = \int_\Omega f\varphi - \int_\Omega \mathbb{K}\nabla R(g_D)\nabla\varphi \quad \forall \varphi \in H^1_0(\Omega)}$$

(4) is the varitional formulation of (3)

- Non-homogeneous Neumann boundary conditions

$$\begin{cases} -\operatorname{div}(\mathbb{K}\nabla u) = f & f \in L^2(\Omega) \\ \text{"}(\mathbb{K}\nabla u)\cdot \underline{n}\,\big|_{\partial\Omega}\text{"} = g_N & g_N \in L^2(\partial\Omega) \end{cases}$$

$u \in H^1(\Omega)$ $\qquad \varphi \in H^1(\Omega)$

$$\int_\Omega -\operatorname{div}(\mathbb{K}\nabla u)\,\varphi = \int_\Omega \mathbb{K}\nabla u\cdot\nabla\varphi - \int_{\partial\Omega}(\mathbb{K}\nabla u)\cdot\underline{n}\,\varphi$$
$$= \int_\Omega \mathbb{K}\nabla u\cdot\nabla\varphi - \int_{\partial\Omega} g_N\,\varphi$$

$$\boxed{\begin{array}{l} \text{Find } u\in H^1(\Omega) \ : \ \int_\Omega u = 0 \qquad\qquad (\mathcal{S}) \\ \int_\Omega \mathbb{K}\nabla u\,\nabla\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g_N\varphi \qquad \varphi\in H^1(\Omega) \end{array}}$$

$\varphi = 1$ $\qquad 0 = \int_\Omega f + \int_{\partial\Omega} g_N$ compatibility condition

<u>theorem</u> :

<u>if</u> $\int_\Omega f + \int_{\partial\Omega} g_N = 0$ then the problem $(\mathcal{S})$ is well posed.

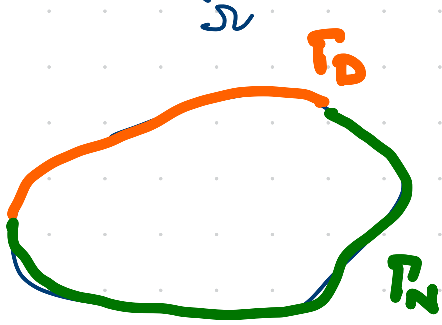
<u>proof</u> : based on Lax-Milgram.

coercivity $\qquad u\in H^1(\Omega) \ : \ \int_\Omega u = 0.$

$$\int_\Omega \mathbb{K}|\nabla u|^2 \geq \beta\int_\Omega |\nabla u|^2 \underset{?}{\geq} \alpha\,\|u\|_1^2$$

Poincaré inequality $\qquad \|u\|_0 \leq C_P'\,\|\nabla u\|_0$
$\qquad \forall u\in H^1(\Omega) : \int_\Omega u = 0.$

- Mixed boundary conditions



$\gamma_{\Gamma_D}(u) = g_D \qquad \mathbb{K}\nabla u\cdot n\big|_{\Gamma_N} = g_N$

$$\begin{cases} -\operatorname{div}(\mathbb{K}\nabla u) = f \\ \gamma_{\Gamma_D}(u) = g_D \qquad \mathbb{K}\nabla u \cdot \underline{n}\big|_{\Gamma_N} = g_N \end{cases}$$

$u = u_0 + R(g_D) \qquad u_0 \in H^1_{0,\Gamma_D}(\Omega) \quad ---$

--- deduce the variational formulation ---

<u>Remark</u> : solutions of the variational formulations are called "weak" solutions.

- they verify the diffusion equation in a suitable weak sense
- if they happen to be regular enough, they verify the diffusion eq. also point wise.

1.5 Galerkin methods

V = Hilbert space $\qquad H_0^1(\Omega),\ H^1(\Omega),\ H^1_{0,\Gamma_D}(\Omega)$

$$\boxed{\text{Find } u \in V \ : \ a(u,\varphi) = \underbrace{F}_{\text{cont. functional on } V}(\varphi) \quad \forall \varphi \in V.} \qquad (6)$$

$V_h \subseteq V$ finite dimensional subspace of V.
$\dim(V_h) = N_h \qquad \lim_{h\to 0} N_h = +\infty$

$$\boxed{\text{Find } u_h \in V_h \ : \ a(u_h, \varphi_h) = F(\varphi_h) \quad \forall \varphi_h \in V_h} \ (7)$$

u_h is the Galerkin approximation of u

(7) is the Galerkin problem associated to (6).

- Lax-Milgram Lemma provides well posedness for (7).
- Galerkin orthogonality

$$a(u,\varphi_h) = F(\varphi_h) \qquad \forall \varphi_h \in V_h$$
$$a(u_h,\varphi_h) = F(\varphi_h) \qquad \forall \varphi_h \in V_h$$

$$\boxed{a(u-u_h,\varphi_h) = 0 \qquad \forall \varphi_h \in V_h}$$

Céa Lemma
Galerkin orthogonality.

- Quasi-optimality of the Galerkin method

$$\alpha \, \|u-u_h\|_1^2 \underset{\substack{\uparrow \\ \text{coercivity}}}{\leq} a(u-u_h, u-u_h)$$

$$\leq a(u-u_h,\ u-u_h+\varphi_h-\varphi_h) =$$

$$= a(u-u_h,\ u-\varphi_h) + \underbrace{\cancel{a(u-u_h,\ \varphi_h-u_h)}}_{=0 \quad \text{GOa}}$$

$$\leq M \cdot \|u-u_h\|_1 \cdot \|u-\varphi_h\|_1$$

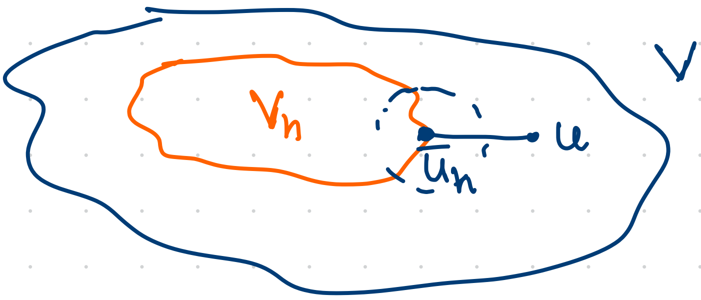
continuity of a

$$\|u-u_h\|_1^{\cancel{2}} \leq \frac{M}{\alpha} \cancel{\|u-u_h\|_1} \cdot \|u-\varphi_h\|_1$$

generic function in V_h

$$\|u-u_h\|_1 \leq \frac{M}{\alpha} \min_{\varphi_h \in V_h} \|u-\varphi_h\|_1$$

Quasi-optimality of the Galerkin methods.



$\bar{u}_h$ the optimal