$u_h \in V_h = \{ \underline{u}_h \in H^1_0(\Omega)^2 : \underline{u}_h|_T \in \mathbb{P}^1(T)^2 \ \forall T \in \mathcal{T}_h \}$.

$\operatorname{div} \underline{u}_h|_T \in \mathbb{P}^0(T)$

$Q_h = \{ q_h \in L^2(\Omega) : \int_\Omega q_h = 0 \quad q_h \in \mathbb{P}^0(T) \ \forall T \in \mathcal{T}_h \}$.

$\# V_h = 2 \cdot \#$ Internal vertices. $\quad \dim(Q_h) > \dim(V_h)$

$\# Q_h = \#$ Elements. -1

the solution $\underline{u}_h$ $\quad \int q_h \cdot \operatorname{div} \underline{u}_h = 0 \quad \forall q_h \in Q_h$

$\underline{u}_h \in \operatorname{Ker} B_h = \{ \underline{v}_h : \int \operatorname{div} \underline{v}_h \, q_h = 0 \ \forall q_h \}$.

↳ Ker B_h needs to be big enough to have approximation properties!

this is not the case for the $\mathbb{P}^1$ - $\mathbb{P}_0$ choice!.

When Ker B_h does not have good approximation properties, the discrete solution may show "locking".

let's find a fix:

$$V_h^2 = \{ \underline{v}_h \in H_0^1(\Omega)^2 : \underline{v}_h|_T \in (\mathbb{P}^2(T))^2 \quad \forall T \in \mathcal{T}_h \}.$$

$$Q_h = \{ q_h \in L^2(\Omega) : \int_\Omega q_h = 0 \quad q_h|_T \in \mathbb{P}^0(T) \quad \forall T \in \mathcal{T}_h \}$$

↳ this choice guarantees solvability.

Full rank condition:

$$\forall q_h \quad \exists \underline{u}_h \quad : \quad \int \mathrm{div}\, \underline{u}_h \cdot q_h > 0.$$

$\mathrm{div} : H_0^1(\Omega)^2 \to L_0^2(\Omega)$ is surjective (functional analysis)

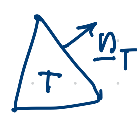
$q_h \in Q_h \qquad \exists \; \underline{u} \in H_0^1(\Omega)^2 \quad : \quad \mathrm{div}\, \underline{u} = q_h.$

$$\int \mathrm{div}\, \underline{u} \cdot q_h = \int |q_h|^2$$

we construct a $\Pi \underline{u} \in V_h^2$: $\displaystyle\int_\Omega \mathrm{div}(\Pi \underline{u})\, q_h = \int_\Omega \mathrm{div}\, \underline{u} \cdot q_h \quad \forall q_h.$

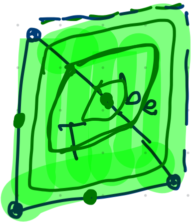
$$\int_T \mathrm{div}(\Pi \underline{u}) = \int_T \mathrm{div}\, \underline{u} \quad \forall T.$$

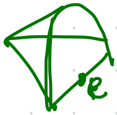
$$\int_T \mathrm{div}(\Pi \underline{u} - \underline{u}) = 0 \qquad \int_{\partial T} (\Pi \underline{u} - \underline{u}) \cdot \underline{n}_T = 0.$$



we construct a $\Pi \underline{u}$ such that

$$\int_e \Pi \underline{u} = \int_e \underline{u} \qquad \forall e, \text{ edge of the mesh}$$





$\Pi \underline{u}$ is constructed only using the edge bubbles, i.e., the dof •

$$\Pi \underline{u} = \sum_{\substack{e \\ \text{edge of } \mathcal{T}_h}} \underline{\alpha}_e \, b_e \qquad \int_e b_e = 1$$

$$\int_{\bar{e}} \Pi \underline{u} = \int \underline{\alpha}_{\bar{e}} \cdot b_{\bar{e}} = \underline{\alpha}_{\bar{e}} \qquad \forall \bar{e} \in \mathcal{T}_h$$

$$\Rightarrow \quad \underline{\alpha}_e = \int_e \underline{u}$$ for all edges of the mesh

$$\Pi \underline{u} = \sum_e \left(\int_e \underline{u} \right) b_e$$ is what I'm looking for.

$$\Rightarrow \quad \int \operatorname{div} \Pi \underline{u} \; q_h = \int q_h^2 > 0 \qquad \forall q_h \in Q_h \;\; q_h \neq 0.$$

$$\ker B_h = \{ \underline{v}_h \in V_h^2 : \int q_h \cdot \operatorname{div} \underline{v}_h = 0 \;\; \forall q_h \in Q_h \}$$

I would need to prove that it has good approximation properties.

→ One could prove that this is true.

Stabilisation techniques

if I have only $\mathbb{P}_1$ continuous finite elements...

$q_h \neq 0$ $\quad \int q_h \cdot \operatorname{div} \underline{v}_h = 0 \quad \forall \underline{v}_h$ spurious mode.

2° equation:

$$\int \operatorname{div} \underline{u}_h \, q_h = 0 \qquad \forall q_h \in Q_h$$

weaken the constraint... design a g_h

$$\int \operatorname{div} \underline{u}_h \cdot q_h = \int g_h \, q_h$$

- g_h should be small enough. $g_h \to 0$ $h \to 0$.
- g_h should be "big" on the spurious mode.

Pitkaranta '85

$$\text{"}\int\text{"} g_h q_h = \langle g_h, q_h \rangle = \boxed{\alpha \sum_K h_K^2 \int_K \nabla p_h \cdot \nabla q_h}$$

diffusion term on the pressure.

$$\int_\Omega \nabla \underline{u}_h \cdot \nabla \underline{v}_h + \int_\Omega p_h \operatorname{div} \underline{v}_h = \int f \, v_h \qquad 1^{\circ}\text{eq}$$

$$\int_\Omega \operatorname{div} \underline{u}_h \, q_h - \alpha \sum_K h_K^2 \int_K \nabla p_h \cdot \nabla q_h = 0. \qquad 2^{\circ}\text{eq}$$

$\underline{v}_h = \underline{u}_h \quad q_h = p_h \qquad 1^{\circ}\text{eq} - 2^{\circ}\text{eq}.$

$$\int |\nabla \underline{u}_h|^2 + \alpha \sum h_K^2 \int_K |\nabla p_h|^2$$

this is a norm!

$$\alpha h^2 \int_\Omega |\nabla p_h|^2$$

the problem is elliptic now !!

→ stability is true in this norm.

but we do not know if we have approximation properties ...

- the stabilization is NOT consistent

(u, p) solution of the continuous problem do not verify the discrete one.

Consistent stabilization as SUPG

$$\int \operatorname{div} \underline{u}_h q_h = \alpha \sum_K h_K^2 \int_K (\Delta_K \underline{u}_h + \nabla p_h - \underline{f}) \nabla q_h$$

Δ_K Laplace element by element

this is consistent, there are good choices of α that gives solvability and approximation.

MATRIX STRUCTURE

Pitkaranta

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}$$

C is the matrix corresponding to $-\alpha \sum_K h_K^2 \int \nabla p_h \nabla q_h$

SUPG

$$\begin{pmatrix} A & B^T \\ \tilde{B} & -C \end{pmatrix}$$

$\tilde{B}$: $\int \operatorname{div} \underline{u}_h q_h - \alpha \sum_K h_K^2 \int_K \Delta_K \underline{u}_h \nabla q_h$

C: as before

SUPG works for good choice of the parameter α and provide good (low order) approximation.