$$\int_\Omega \frac{u_h^{j+1}-u_h^j}{\Delta t}\, v_h + \int_\Omega \nabla\left(\theta u_h^{j+1} + (1-\theta) u_h^j\right)\nabla v_h = \int_\Omega \left(\theta f^{j+1} + (1-\theta) f^j\right) v_h \qquad (1)$$

$\theta = 0$ explicit Euler

$\theta = 1$ implicit Euler

$\theta = 1/2$ C.N. method.

Algebraic identity

$$\int_\Omega (u_h^{j+1} - u_h^j)\left[\theta u_h^{j+1} + (1-\theta) u_h^j\right] = \frac{1}{2}\|u_h^{j+1}\|_{L^2}^2 - \frac{1}{2}\|u_h^j\|_{L^2}^2 + \left(\theta - \frac{1}{2}\right)\|u_h^{j+1} - u_h^j\|_{L^2}^2$$

$\theta = 1$ Implicit Euler. $v_h = u_h^{j+1}$

$$\left(\frac{1}{\Delta t}\boxed{\underbrace{\int_\Omega (u_h^{j+1} - u_h^j)\, u_h^{j+1}}_{\text{algebraic. id.}}} + \boxed{\underbrace{\int_\Omega \nabla u_h^{j+1} \nabla u_h^{j+1}}_{\text{coercivity.}}} = \int_\Omega f u_h^{j+1}\right) * \Delta t$$

$$\boxed{\frac{1}{2}\|u_h^{j+1}\|^2 - \frac{1}{2}\|u_h^j\| + \frac{1}{2}\|u_h^{j+1} - u_h^j\|^2} + \boxed{\Delta t \int_\Omega |\nabla u_h^{j+1}|^2}$$

Then:

$$\frac{1}{2}\|u_h^{j+1}\|^2 - \frac{1}{2}\|u_h^j\|^2 + \boxed{\Delta t\, \alpha \|u_h^{j+1}\|_{H^1}^2} \le$$

$$\le \Delta t \int_\Omega f \cdot u_h^{j+1} \le \Delta t \|f^{j+1}\| \cdot \|u_h^{j+1}\|_{L^2}$$

$$\le \Delta t \|f^{j+1}\|\, \|u_h^{j+1}\|_{H^1}$$

$$\le \frac{1}{2}\frac{\Delta t}{\alpha}\|f^{j+1}\|^2 + \boxed{\frac{1}{2}\alpha \Delta t \|u_h^{j+1}\|_{H^1}^2}$$

$$2ab \le \frac{1}{2}\frac{a^2}{\alpha} + \frac{1}{2}\alpha b^2$$

$$\sum_{j=0}^{N-1}\left[\cancel{\tfrac{1}{2}}\|u_h^{j+1}\|^2-\cancel{\tfrac{1}{2}}\|u_h^j\|^2+\cancel{\tfrac{1}{2}}\Delta t\,\alpha\,\|u_h^{j+1}\|^2_{H^1}\le\cancel{\tfrac{1}{2}}\frac{\Delta t}{\alpha}\|f^{j+1}\|^2\right]$$

$$\underbrace{\sum_{j=0}^{N-1}\left(\|u_h^{j+1}\|^2-\|u_h^j\|^2\right)}_{\substack{\text{telescopic}\\ \| \\ \|u_h^N\|^2-\|u_h^0\|^2}}+\alpha\sum_{j=0}^{N-1}\Delta t\,\|u_h^{j+1}\|^2_{H^1}\le\frac{1}{\alpha}\sum_{j=0}^{N-1}\Delta t\,\|f^{j+1}\|^2$$

$$\underbrace{\|u_h^N\|^2}_{L^\infty(L^2)}+\alpha\underbrace{\sum\Delta t\,\|u_h^{j+1}\|^2_{H^1}}_{L^2(H^1)}\le\frac{1}{\alpha}\sum_j\Delta t\,\|f^{j+1}\|^2+\|u_h^0\|^2$$

$\longrightarrow$ free problem is unconditionally stable!

if we interpret $u_h(x,t)$ as a piecewise constant function in time. Then $\sum\Delta t\,\|u_h^{j+1}\|^2=\int_0^T\|u_h\|^2_{H^1}$

$\underline{\theta=0.}$ $\quad v_h=u_h^{j+1}$

$$\underbrace{\int_\Omega(u_h^{j+1}-u_h^j)\,u_h^{j+1}}_{\mathrm{I}}+\Delta t\underbrace{\int_\Omega\nabla u_h^j\,\nabla u_h^{j+1}}_{\mathrm{II}}=\Delta t\int_\Omega f\,u_h^{j+1}$$

$$\mathrm{I}:\quad\int_\Omega(u_h^{j+1}-u_h^j)\,u_h^{j+1}=\frac{1}{2}\|u_h^{j+1}\|^2-\frac{1}{2}\|u_h^j\|^2+\frac{1}{2}\|u_h^{j+1}-u_h^j\|^2$$

$$\mathrm{II}\quad\int_\Omega\nabla u_h^j\,\nabla u_h^{j+1}=\int_\Omega|\nabla u_h^{j+1}|^2+\int_\Omega\nabla(u_h^j-u_h^{j+1})\,\nabla u_h^{j+1}$$

$$u_h^j=u_h^{j+1}-u_h^{j+1}+u_h^j$$

$$\frac{1}{2}\|u_h^{j+1}\|^2 - \frac{1}{2}\|u_h^j\|^2 + \frac{1}{2}\|u_h^{j+1} - u_h^j\|^2$$

$$+ \Delta t \int_\Omega |\nabla u_h^{j+1}|^2 + \Delta t \int_\Omega \nabla(u_h^j - u_h^{j+1}) \nabla u_h^{j+1}$$

$$= \Delta t \int_\Omega f^j u_h^{j+1}$$

$$\frac{1}{2}\|u_h^{j+1}\|^2 - \frac{1}{2}\|u_h^j\|^2 + \boxed{\frac{1}{2}\|u_h^{j+1} - u_h^j\|^2} + \boxed{\alpha \Delta t \|u_h^{j+1}\|^2_{H^1}}$$

$$\leq \underbrace{\Delta t \int_\Omega f^j u_h^{j+1}}_{A} + \underbrace{\Delta t \int_\Omega \nabla(u_h^{j+1} - u_h^j) \nabla u_h^{j+1}}_{B}$$

$$A \quad \Delta t \int f^j u_h^{j+1} \leq \frac{\Delta t}{\alpha}\|f^j\|^2 + \boxed{\frac{\Delta t\,\alpha}{4}\|u_h^{j+1}\|^2_{H^1}}$$

norms do not match!

$$B \quad \Delta t \int_\Omega \nabla(u_h^{j+1} - u_h^j)\nabla u_h^{j+1} \leq \boxed{\frac{\Delta t}{\alpha}\|\nabla(u_h^{j+1} - u_h^j)\|^2_{L^2}}$$

$$+ \boxed{\frac{\Delta t\,\alpha}{4}\|u_h^{j+1}\|^2_{H^1}}$$

$$\frac{\Delta t}{\alpha}\|\nabla(u_h^{j+1} - u_h^j\|^2_{L^2} \leq \frac{\Delta t}{\alpha} C_{inv} h^{-2} \|u_h^{j+1} - u_h^j\|^2_{L^2}$$

$$\frac{1}{2}\|u_h^{j+1}\|^2 - \frac{1}{2}\|u_h^j\|^2 + \left(\frac{1}{2} - \frac{\Delta t}{\alpha} C_{inv} h^{-2}\right)\|u_h^{j+1} - u_h^j\|^2_{L^2}$$

$$+ \frac{\Delta t}{2}\alpha \|u_h^{j+1}\|^2_{H^1} \leq \frac{\Delta t}{\alpha}\|f^j\|^2_{L^2}$$

Stability condition $\quad \frac{1}{2} - \frac{\Delta t}{\alpha} C_{inv} h^{-2} > 0$

CFL condition.

Courant-Fredrichs-Levy condition.

$$\frac{\Delta t}{\alpha} C_{inv} \cdot h^{-2} < \frac{1}{2} \qquad \Delta t < \frac{1}{2} \frac{\alpha}{C_{inv}} \cdot h^2$$

Explicit Euler:

$$M \underline{y}^{j+1} - M \underline{y}^j + \Delta t K \underline{y}^j = \underline{f}^j$$

$$M \underline{y}^{j+1} = \underline{f}^j + M \underline{y}^j - \Delta t K \underline{y}^j$$

- very easy system to solve because the mass matrix is well-conditioned.
- in the case of $\mathbb{P}_1$ FE, the mass matrix can be approximated with a diagonal matrix.

$\Rightarrow$ mass lumping (see exercise).

APPROXIMATION

$$\|u_h^N\|_0^2 + \alpha \sum_{j=0}^{N-1} \Delta t \, \|u_h^{j+1}\|_{H^1}^2 \leq C \sum_{j=1}^{N} \|f^j\|$$

$$\|\underbrace{u(x,t^N)}_{\text{continuous solution at final time}} - u_n^N\|^2 + \alpha \sum \Delta t \, \|\underbrace{u(x,t^{j+1})} - u_h^{j+1}\|_{H^1}^2$$

continuous solution at final time

$$\leq C_1 (\Delta t)^2 + C_2 h^{2r}$$

$\theta = 0$

$\theta = 1$

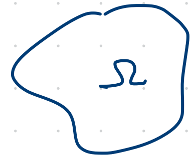
r = order of polynomials in space.

$\theta = \frac{1}{2}$

$$\leq C_1 (\Delta t)^4 + C_2 h^{2r}$$

Chapter 5 Discretization of the wave equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t} - c^2 \Delta u = f & \Omega \times (0,T] \\ u(x,0) = u_0 & \Omega, \ t=0 \\ \dfrac{\partial u}{\partial t}(x,0) = u_1 & \Omega, \ t=0. \\ u(x,t) = g & x \in \partial\Omega \ \ t \in (0,T]. \end{cases}$$



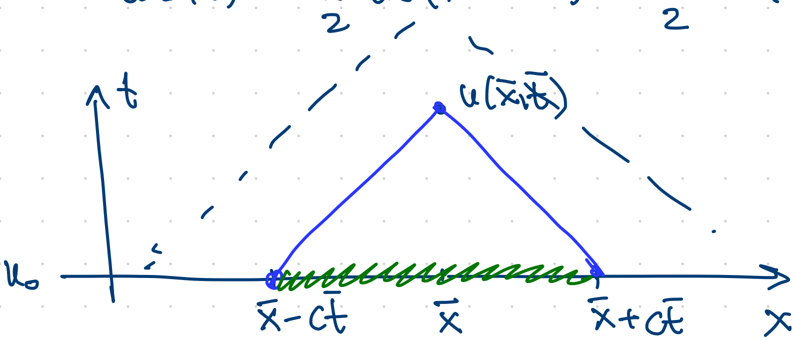
$c^2 \Delta u \to \operatorname{div}(A \nabla u)$ is the medium is not homogeneous.

Remark $\Omega = \mathbb{R}$ $f = 0.$

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0 \\ u(x,0) = u_0 \\ \partial_t u(x,0) = u_1 \end{cases}$$

D'Alambert formula:

$$u(x,t) = \frac{1}{2} u_0(x - ct) + \frac{1}{2} u_0(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy$$



5.1 Conservation of energy (for $f=0$.)

$v \in H_0^1(\Omega)$

$$\int_\Omega \partial_{tt} u \, v + \int_\Omega A \nabla u \cdot \nabla v = \cancel{\int_\Omega f v} \qquad f=0.$$

if u is regular enough .... $v = \partial_t u$

$$\underbrace{\int_\Omega \partial_{tt} u \cdot \partial_t u}_{\frac{1}{2} \partial_t \left(|\partial_t u|^2 \right)} + \underbrace{\int A \nabla u \cdot \nabla \partial_t u}_{\frac{1}{2} \partial_t \left(A \nabla u \cdot \nabla u \right)} = 0$$

$$\frac{1}{2} \partial_t \Big[\underbrace{\int_\Omega |\partial_t u|^2 + \int_\Omega A \nabla u \cdot \nabla u}_{E} \Big] = 0.$$

this quantity is constant and it's the energy!

$\Rightarrow$ the energy is conserved $\qquad \frac{\partial}{\partial t} E = 0$

$$\int_0^T \frac{\partial}{\partial t} E \, dt = E(T) - E(0) = 0.$$

$$\int_\Omega \left(\partial_t u(x,T) \right)^2 + \int_\Omega A \nabla u(x,T) \nabla u(x,T)$$

$$= \int_\Omega (u_1)^2 + \int_\Omega A \nabla u_0 \nabla u_0$$

$$A |\nabla u_0|^2$$

5.2 Wellposedness of free wave equation. (Evans 7.2)

- uniqueness. u_1, u_2

$$\partial_t (u_1 - u_2) = 0 \qquad \nabla \cdot (u_1 - u_2) = 0. \qquad \to u_1 = u_2$$

- existence $\qquad H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$

via Galerkin method

$$V_N = \text{span}\{\varphi_1 \dots \varphi_N\} \qquad \{\varphi_i\}_{i\in\mathbb{N}} \text{ basis for } H_0^1(\Omega).$$

$$u = \sum u^j \varphi^j \qquad \underline{U} = \{u^j\}_{j=1}^N$$

$$M\ddot{\underline{U}} + A\underline{U} = \underline{F} \qquad M_{ij} = \int \varphi_i \varphi_j$$

$$\underline{U}(0) = \underline{U}_0 \qquad A_{ij} = \int \mathbb{A} \nabla\varphi_i \nabla\varphi_j$$

$$\dot{\underline{U}}(0) = \underline{U}_1 \qquad F_j = \int f \varphi_j$$

$$\int_\Omega |\partial_t u_N|^2 + \int_\Omega \mathbb{A} \nabla u_N \nabla u_N$$ is bounded. (equal to the initial energy).

$$u_N \xrightarrow{?} u$$ in which sense u_N converges to u is slightly delicate ... (Evans).