Series 9 - November 20, 2024

Exercise 1.

1) Show that the following nonlinear SDE

$$dX_t = dt + 2\sqrt{X_t}dW_t \tag{1.1}$$

has solution $X_t = (W_t + \sqrt{X_0})^2$ for $t \leqslant \tau = \inf\{t > 0 \,:\, W_t < -\sqrt{X_0}\}$.

2) Show that the following nonlinear SDE

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1 + X_t^2}dW_t \tag{1.2}$$

has solution $X(t) = \sinh(\operatorname{arcsinh}(X_0) + t + W_t)$.

Solution

1) We consider the candidate solution $\hat{X}_t = (W_t + \sqrt{X_0})^2$ and apply Itô lemma to X_t. Calling $f(t, x) = (x + \sqrt{X_0})^2$, one gets $\frac{\partial f}{\partial t}(t, x) = 0$, $\frac{\partial f}{\partial x}(t, x) = 2(x + \sqrt{X_0})$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = 2$. Therefore, observing that $Y_t = W_t$ is also an Itô process ($dY_t = dW_t$), it follows that

$$\begin{aligned} df(t, Y_t) &= \left(\frac{\partial f(t, W_t)}{\partial t} + \frac{\partial f(t, W_t)}{\partial x} \cdot 0 + \frac{1}{2}\frac{\partial^2 f(t, W_t)}{\partial x^2} \cdot 1^2\right)dt + \frac{\partial f(t, W)}{\partial x} \cdot 1dW_t \\ &= \left(2 \cdot \frac{1}{2}\right)dt + 2(W_t + \sqrt{X_0})dW_t \\ &= dt + 2\sqrt{(W_t + \sqrt{X_0})^2}dW_t, \quad t \leqslant \tau. \end{aligned} \tag{1.3}$$

and therefore

$$\begin{aligned} d\hat{X}_t = df(t, Y_t) &= dt + 2\sqrt{(W_t + \sqrt{X_0})^2}dW_t \\ &= dt + 2\sqrt{\hat{X}_t}dW_t, \quad t \leqslant \tau. \end{aligned} \tag{1.4}$$

Therefore, $X(t) = (W(t) + \sqrt{X_0})^2$ is solution of (1.1) up to $t = \tau$.

2) Again, we apply Itô lemma to $X(t)$. Calling $f(t, x) = \sinh(C + t + x)$ with C constant, one gets $\frac{\partial f}{\partial t}(t, x) = \frac{\partial f}{\partial x}(t, x) = \cosh(C + t + x)$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = \sinh(C + t + x)$. Therefore, one obtains

$$\begin{aligned} df(t, Y_t) &= \left(\frac{\partial f(t, W_t)}{\partial t} + \frac{\partial f(t, W_t)}{\partial x} \cdot 0 + \frac{1}{2}\frac{\partial^2 f(t, W_t)}{\partial x^2} \cdot 1^2\right)dt + \frac{\partial f(t, W)}{\partial x} \cdot 1dW_t \\ &= \left(\cosh(C + t + W_t) + \frac{1}{2}\sinh(C + t + W_t)\right)dt + \cosh(C + t + W_t)dW_t \\ &= \left(\sqrt{1 + \sinh(C + t + W_t)^2} + \frac{1}{2}\sinh(C + t + W_t)\right)dt + \sqrt{1 + \sinh(C + t + W_t)^2}dW_t \end{aligned} \tag{1.5}$$

and hence

$$dX_t = df(t,Y_t) = \left(\sqrt{1+\sinh(C+t+W_t)^2} + \frac{1}{2}\sinh(C+t+W_t)\right)dt + \sqrt{1+\sinh(C+t+W_t)^2}dW_t$$
$$= \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dW_t \tag{1.6}$$

Therefore, $X(t) = \sinh(C+t+W_t)$ is solution of (1.2). To determine C, notice that $X(0) = \sinh(C)$, hence $C = \text{arcsinh}(X_0)$.

Exercise 2.

Consider the modified Euler–Maruyama method given by

$$X_{n+1} = X_n + f(t_n, X_n)h + g(t_n, X_n)\xi_n,$$

where $\{\xi_n\}_{n\geqslant 0}$ is a sequence of independent random variables such that ξ_n is independent of X_n for all n and

$$\mathbb{E}[\xi_n] = 0, \qquad \mathbb{E}[\xi_n^2] = h, \qquad \mathbb{E}[\xi_n^3] = 0, \qquad |\mathbb{E}[\xi_n^4]| = o(h^2).$$

Since $\mathbb{E}[\xi_n^\ell] = \mathbb{E}[\varDelta W_n^\ell]$ for $\ell = 1,2,3$ and $\mathbb{E}[\xi_n^4]$ is bounded, then it is possible to prove that this method has weak order 1.

i) Give an example of discrete random variables ξ_n satisfying the hypotheses above.

ii) Verify numerically that this method has weak order 1. Set $f(x) = \lambda x$ with $\lambda = 2$, $g(x) = \mu x$ with $\mu = 0.1$, $X_0 = 1$ and $T = 1$. Choose different step sizes $h = 2^{-i}$ with $i = 4, \dots, 10$ and approximate the expectations of the weak error via using $M = 10^4$ realizations of $\{\xi_n\}_{n\geqslant 0}$.

Solution

An example of discrete random variable ξ_n satisfying the requirements is

$$P(\xi_n = \sqrt{h}) = P(\xi_n = -\sqrt{h}) = 1/2,$$

and the plot of the weak order employing this random variable is given in Figure 1.

Exercise 3.

Consider the following geometric Brownian motion

$$\begin{aligned} dX_t = \mu X_t dt + \sigma X_t dW_t,\ t \in [0,T], \\ X(0) = X_0 \in \mathbb{R} \end{aligned} \tag{3.1}$$

with $\mu, \sigma \in \mathbb{R}$. Consider a uniform partition $P = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of size $\varDelta t$. For the linear SDE (3.1), the Euler-Maruyama method produces the recurrence

$$Y_{n+1} = (1 + \mu\varDelta t + \sigma\varDelta W_n)Y_n, \quad n = 0, \cdots, N-1. \tag{3.2}$$

1) Noting that $\varDelta W_n$ is independent of Y_n, take expected values to show that

$$\mathbb{E}[Y_{n+1}] = (1+\mu\varDelta t)\mathbb{E}[Y_n].$$

Considering the limit $\varDelta t \to 0$ and $N \to \infty$ with $N\varDelta t = T$ fixed, show that

$$\mathbb{E}[Y_N] \to e^{\mu T}\mathbb{E}[X_0],\ N \to \infty$$

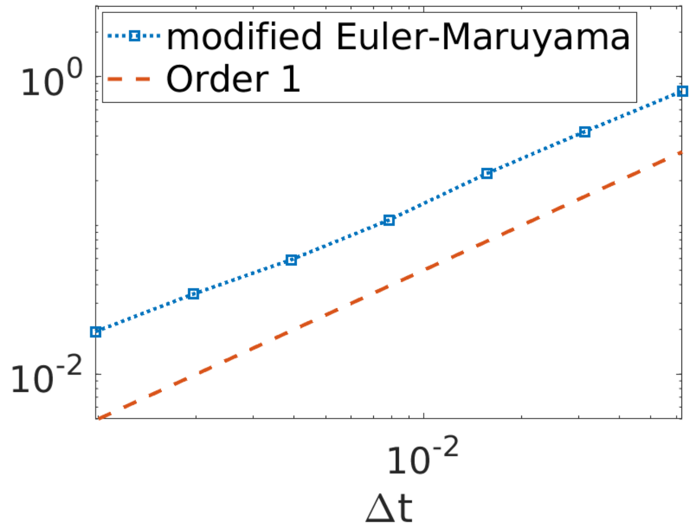


Figure 1: Weak order of convergence of the modified Euler–Maruyama method of Exercise 2.

2) Show that

$$\mathbb{E}[Y_{n+1}^2] = ((1+\mu\Delta t)^2 + \sigma\Delta t)\mathbb{E}[Y_n^2]$$

and, hence under $\Delta t \to 0$ and $N \to \infty$ with $N\Delta t = T$, one has that

$$\mathbb{E}[Y_N^2] \to e^{(2\mu+\sigma^2)T}\mathbb{E}[X_0], \ \ N \to \infty.$$

3) For $\mu = 2$, $\sigma = 0.1$, $X_0 = 1$, $T = 1$, $M = 10^4$, plot the weak error $|\mathbb{E}[Y_N] - \mathbb{E}[X_T]|$.

Solution

1) As $\mathbb{E}[\Delta W_n] = 0$ and by the property of independence of Brownian increments, one has

$$\mathbb{E}[Y_{n+1}] = \mathbb{E}[(1+\mu\Delta t + \sigma\Delta W_n)]\mathbb{E}[Y_n] = (1+\mu\Delta t)\mathbb{E}[Y_n], \tag{3.3}$$

hence,

$$\mathbb{E}[Y_N] \quad = (1+\mu\frac{T}{N})^N\mathbb{E}[X_0]. \tag{3.4}$$

Then, conclusion follows by taking the limit for $n \to \infty$.

2) Squaring (3.2) and taking the expectation we get

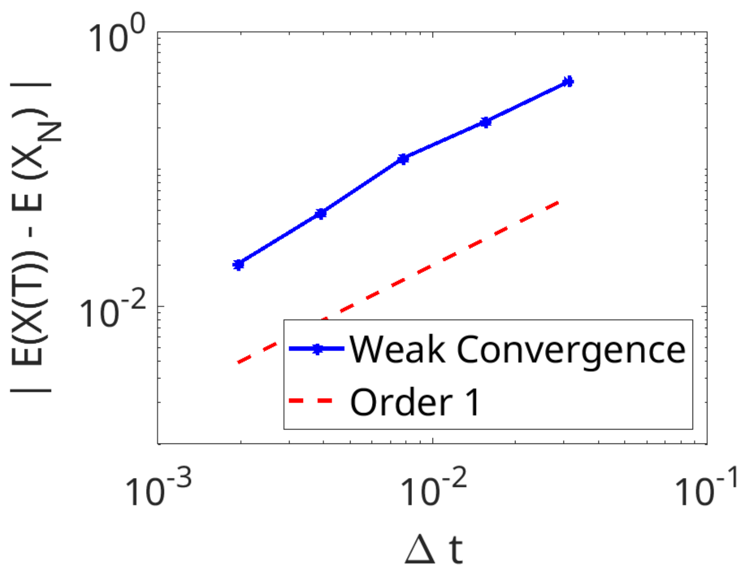
$$\begin{aligned}\mathbb{E}[Y_{n+1}^2] &= \mathbb{E}[(1+\mu\Delta t+\sigma\Delta W_n)^2]\mathbb{E}[Y_n^2] \\ &= \mathbb{E}[(1+\mu\Delta t)^2 + 2(1+\mu\Delta t)(\sigma\Delta W_n) + (\sigma\Delta W_n)^2]\mathbb{E}[Y_n^2] \\ &= ((1+\mu\Delta t)^2 + \sigma^2\Delta t)\mathbb{E}[Y_n^2] \\ &= ((1+\mu\Delta t)^2 + \sigma^2\Delta t)^n\mathbb{E}[X_0^2] \\ &= ((1+(2\mu+\sigma^2)\Delta t + \mu^2(\Delta t)^2)^n\mathbb{E}[X_0^2].\end{aligned} \tag{3.5}$$

Therefore

$$\mathbb{E}[Y_N^2] = \quad = ((1+(2\mu+\sigma^2)\Delta t + \mu^2(\Delta t)^2)^n\mathbb{E}[X_0^2]. \tag{3.6}$$

and considering $N = \frac{T}{\Delta t}$, one gets

$$\lim_{N\to\infty}\mathbb{E}[Y_N^2] = \lim_{N\to\infty}((1+(2\mu+\sigma^2)\frac{T}{N} + \mu^2(\frac{T}{N})^2)^n \quad = \lim_{N\to\infty} e^{(2\mu+\sigma^2)T+\mu^2(\frac{T}{N})T}\mathbb{E}[X_0^2] = e^{(2\mu+\sigma^2)T}\mathbb{E}[X_0^2]. \tag{3.7}$$



3)

Figure 2: Weak convergence for the EM method applied to (3.1).

Exercise 4.

Consider the following Langevin dynamics

$$\begin{aligned} dX_t = \mu X_t dt + \sigma dW_t,\ t \in [0,T], \\ X(0) = X_0 \in \mathbb{R} \end{aligned} \tag{4.1}$$

with $\mu, \sigma \in \mathbb{R}$. Consider a uniform partition $P = \{0 = t_0 < t_1 < ... < t_N = T\}$ of size Δt. For the linear SDE (4.1), the Euler-Maruyama method produces the following recurrence

$$Y_{n+1} = (1 - \mu \Delta t) Y_n + \sigma \Delta W_n. \tag{4.2}$$

Repeat the same computations of Exercise 3.

Solution

The Euler-Maruyama method (4.2), passing to the expectation, after recursion, yields

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}[Y_N] &= \lim_{N\to\infty} (1 - \mu \frac{T}{N})^N \mathbb{E}[X_0] \\ &= e^{-\mu T} \mathbb{E}[X_0] \end{aligned} \tag{4.3}$$

Squaring both sides of (4.2) and taking expectations, we get

$$\mathbb{E}[Y_{n+1}^2] = (1 - \mu \Delta t)^2 \mathbb{E}[Y_n^2] + \sigma^2 \Delta t.$$

A recursion yields

$$\mathbb{E}[Y_N^2] = (1 - \mu \Delta t)^{2N} \mathbb{E}[X_0] + \sigma^2 \Delta t \sum_{j=0}^{N-1} ((1 - \mu \Delta t)^2)^j. \tag{4.4}$$

Under the hypothesis $|1 - \mu \Delta t| < 1$, the second member on the right of (4.4) is a convergent geometric series. Therefore, for $N \to \infty$, we have

$$\mathbb{E}[Y_N^2] \to e^{-2\mu T} \mathbb{E}[X_0^2] + \frac{\sigma^2}{\mu(2 - \mu \Delta t)}.$$

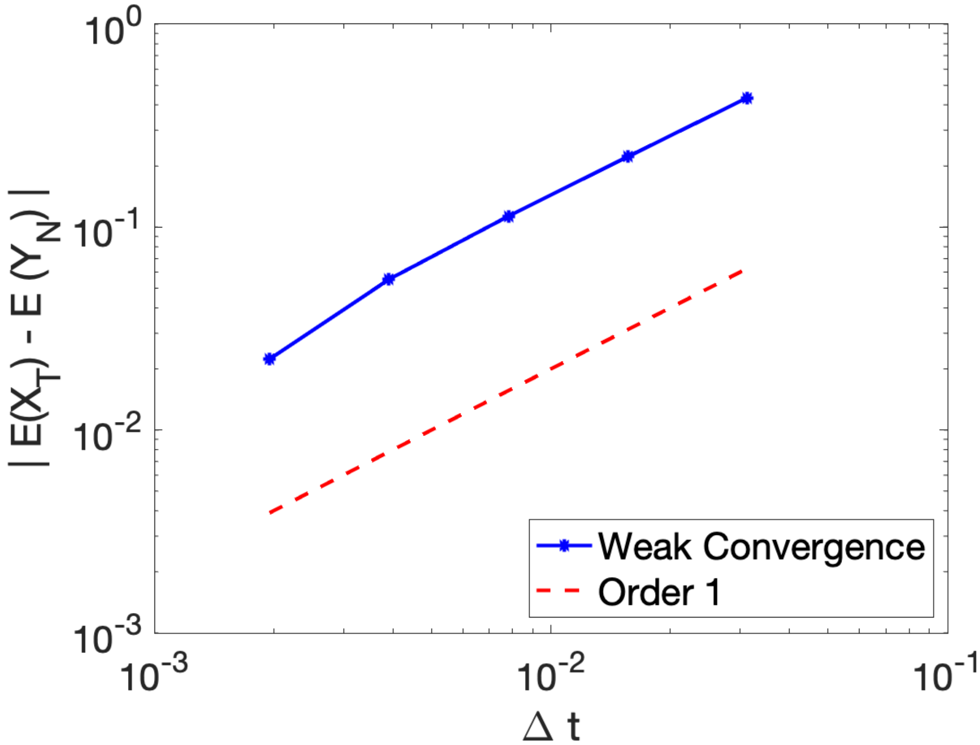


Figure 3: Weak convergence for the EM method applied to (4.2).

Concerning the numerical simulation, the results are shown in Figure 3.

Exercise 5.

Let us recall that if we consider an Itô stochastic differential equation (here one dimensional for simplicity)

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

whose solution is X_t, then, X_t solves the following Stratonovich SDE

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t,$$

where $\underline{a}(t, x) = a(t, x) - \frac{1}{2}b(t, x)\frac{\partial b}{\partial x}(t, x)$. On the contrary, given a Stratonovich SDE

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t,$$

then the corresponding Itô differential equation is

$$dX_t = \left(\underline{a}(t, x) + \frac{1}{2}b(t, x)\frac{\partial b}{\partial x}(t, x)\right)dt + b(t, X_t)dW_t.$$

Let $\lambda, \mu \in \mathbb{R}$ and consider the SDE for $t \in [0, T]$

$$\begin{aligned} dX(t) &= \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \\ X(0) &= X_0. \end{aligned} \tag{5.1}$$

The solution of (5.1) in the Itô sense is given by $X(t) = X_0 e^{\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)}$.

Let $P = \{0 = t_0 < t_1 < ... < t_m = T\}$ be a partition of $[0, T]$ of size Δt and define the Euler polygonal interpolant $\widehat{W}$ of W on P as

$$\widehat{W}(t) = W(t_{n-1}) + (W(t_n) - W(t_{n-1}))\frac{t - t_{n-1}}{t_n - t_{n-1}}, \quad 1 \leqslant n \leqslant m, \quad t_{n-1} \leqslant t \leqslant t_n.$$

If we replace W by $\widehat{W}$ in the SDE (5.1), we obtain the ordinary differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\widehat{X}^m(t) &= \lambda \widehat{X}^m(t) + \mu \widehat{X}^m(t)\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}(t), \\ \widehat{X}^m(0) &= X_0. \end{aligned} \tag{5.2}$$

iv) Compute the solution $\widehat{X}^m(t)$ of (5.2).

v) What is the limit in $\mathrm{L}^2(\Omega)$ of $\widehat{X}^m(t)$ as $m \to \infty$ (i.e., as $\Delta t \to 0$)?

In order to approximate numerically the solution $\widehat{X}^m$ of (5.2) we can use the following scheme

$$\widehat{X}^m(t_{n+1}) = \widehat{X}^m(t_n) + \lambda \widehat{X}^m(t_n)\Delta t + \mu(W(t_{n+1}) - W(t_n))\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \widehat{X}^m(s)\mathrm{d}s, \tag{5.3}$$

where we need to approximate the integral $\int_{t_n}^{t_{n+1}} \widehat{X}^m(s)\mathrm{d}s$.

vi) If we approximate the integral with the Euler formula

$$\int_{t_n}^{t_{n+1}} \widehat{X}^m(s)\mathrm{d}s \approx \Delta t \widehat{X}^m(t_n),$$

what method do we obtain?

vii) We now approximate the integral with the trapezoidal rule

$$\int_{t_n}^{t_{n+1}} \widehat{X}^m(s)\mathrm{d}s \approx \frac{\Delta t}{2}(\widehat{X}^m(t_n) + \widehat{X}^m(t_{n+1})),$$

and make an Euler prediction for the implicit term

$$\widehat{X}^m(t_{n+1}) \approx \widehat{X}^m(t_n) + \lambda \widehat{X}^m(t_n)\Delta t + \mu \widehat{X}^m(t_n)(W(t_{n+1}) - W(t_n)).$$

Write the method derived from these approximations.

viii) Let $T = 1$, $\lambda = 2$, $\mu = 1$, $X_0 = 1$ and consider a uniform partition of $[0, T]$ with $\Delta t = 10^{-2}$. Implement the numerical methods derived in points *vi)* and *vii)*. What solutions do these methods converge to? What are the strong orders of convergence of the methods? In order to observe numerically the strong order, plot the error for different values of $\Delta t = 2^{-i}$ with $i = 4, 5, \dots, 11$ employing $M = 10^4$ different Brownian paths.

Solution

i) We have almost everywhere that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}(t) = \sum_{i=1}^{m} \frac{W(t_i) - W(t_{i-1})}{t_i - t_{i-1}} \chi_{(t_{i-1}, t_i)}(t),$$

and hence $y^m(t) = \widehat{X}^m(t)$ satisfies the ODE

$$\begin{aligned} \dot{y}^m(t) &= f^m(t) y^m(t), \\ y(0) &= X_0, \end{aligned} \qquad \text{where} \quad f^m(t) = \lambda + \mu \sum_{i=1}^{m} \frac{W(t_i) - W(t_{i-1})}{t_i - t_{i-1}} \chi_{(t_{i-1}, t_i)}(t),$$

whose solution is given by $y(t) = X_0 e^{\int_0^t f^m(s)\mathrm{d}s}$. For $t \in [0,T]$, we set k such that $t_{k-1} \leqslant t < t_k$ and compute

$$\begin{aligned}\int_0^t f^m(s)\mathrm{d}s &= \lambda t + \mu \sum_{i=1}^{k-1}(W(t_i) - W(t_{i-1})) + \mu(W(t_k) - W(t_{k-1}))\frac{t - t_{k-1}}{t_k - t_{k-1}} \\ &= \lambda t + \mu\Big(W(t_{k-1}) + (W(t_k) - W(t_{k-1}))\frac{t - t_{k-1}}{t_k - t_{k-1}}\Big) \\ &= \lambda t + \mu \widehat{W}(t).\end{aligned}$$

ii) As $t \mapsto W(t)$ is almost surely continuous, we have for all t

$$\lim_{m\to\infty} \int_0^t f^m(s)\mathrm{d}s = \lambda t + \mu W(t) \text{ in } \mathrm{L}^2(\Omega),$$

and consequently $y^m(t) = \widehat{X}^m(t)$ converges to $\widehat{X}(t) = e^{\lambda t + \mu W(t)}$ in $\mathrm{L}^2(\Omega)$, which is in fact the solution of (5.1) in the Stratonovich sense. That makes sense because using $\widehat{W}(t)$ is anticipating.

iii) We obtain the Euler–Maruyama method from Series 1.

iv) The method can be explicitly written as

$$\widehat{X}^m_{n+1} = \widehat{X}^m_n + \lambda \widehat{X}^m_n \Delta t + \mu \Delta W_{n+1} \widehat{X}^m_n + \tfrac{1}{2}\mu \Delta W_{n+1}(\lambda \widehat{X}^m_n \Delta t + \mu \widehat{X}^m_n \Delta W_{n+1}),$$

where $\widehat{X}^m_n = \widehat{X}^m(t_n)$ and $\Delta W_{n+1} = W(t_{n+1}) - W(t_n)$.

v) The Euler–Maruyama method converges with strong order 1/2 to the Itô solution $X(t)$. Moreover, we verify that the second method converges to the solution $\widehat{X}(t) = e^{\lambda t + \mu W(t)}$ in the Stratonovich sense in $\mathrm{L}^2(\Omega)$ and that the strong order of convergence is 1. The plots are given in Figure 4.

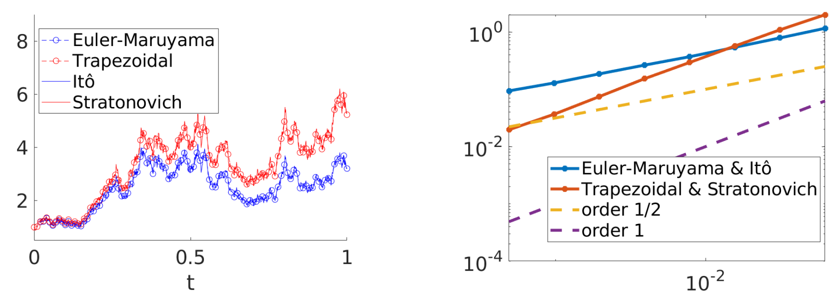


Figure 4: Euler–Maruyama and trapezoidal methods employed in Exercise 5 with their rates of convergence.