Series 4 - October 9, 2024

Exercise 1.

Let $G \in \mathrm{M}^2(0,T)$ with $t \mapsto G(t,\omega)$ continuous for almost every ω and consider a partition $P_\Delta = \{0 = t_0 < t_1 < \ldots < t_{m_\Delta} = T\}$ of $[0,T]$ of size Δ, i.e., $t_j = j\Delta$ for $j = 0,\ldots,m_\Delta$. Assume that $\mathbb{E}[G(t)G(s)]$ is a continuous function of t and s. Show that

$$\lim_{\Delta\to 0}\sum_{j=1}^{m_\Delta} G(t_{j-1})(W(t_j)-W(t_{j-1})) = \int_0^T G(t)\,\mathrm{d}W(t) \quad \text{in } \mathrm{L}^2.$$

Solution

From G define the step process $G_m(r,\omega) = \sum_{j=1}^m G(t_{j-1},\omega)\chi_{[t_{j-1},t_j[}(r)$ and define

$$I_m := \mathbb{E}\left[\left(\sum_{j=1}^{m_\Delta} G(t_{j-1})(W(t_j)-W(t_{j-1})) - \int_0^T G(t)\mathrm{d}W(t)\right)^2\right]. \tag{1.1}$$

Then, by the Itô isometry we obtain

$$I_m = \mathbb{E}\left[\left(\int_0^T (G_m(t)-G(t))\mathrm{d}W(t)\right)^2\right] = \int_0^T \mathbb{E}[(G_m(t)-G(t))^2]\mathrm{d}t. \tag{1.2}$$

Due to the assumption we have

$$\lim_{s\to t}\mathbb{E}[(G(s)-G(t))^2] = 0, \tag{1.3}$$

which implies for all $t \in [0,T]$

$$\lim_{m\to\infty}\mathbb{E}[(G_m(t)-G(t))^2] = 0. \tag{1.4}$$

Moreover, we also have

$$E[(G_m(t)-G(t))^2] \leqslant 2(\mathbb{E}[G_m(t)^2] + \mathbb{E}[G(t)^2]) \leqslant 4\max_{t\in[0,T]}\mathbb{E}[G(t)^2], \tag{1.5}$$

where the right-hand side is finite by assumption. Therefore, applying the dominated convergence theorem we deduce that $I_m \to 0$ as $m \to \infty$, which concludes the proof.

Exercise 2.

Let $(W(t), t \geqslant 0)$ be a one-dimensional Brownian motion. Without implement the Itô formula but using the construction of the stochastic integral, show that:

i) $\mathrm{d}(W^2) = 2W\mathrm{d}W + \mathrm{d}t$,

ii) $\mathrm{d}(tW) = W\mathrm{d}t + t\mathrm{d}W$.

Solution

i) We already know that

$$\int_0^T W(t)\mathrm{d}W(t) = \tfrac{W(T)^2}{2} - \tfrac{T}{2},$$

and hence

$$\int_r^s W(t)\mathrm{d}W(t) = \tfrac{W(s)^2 - W(r)^2}{2} - \tfrac{s-r}{2},$$

which gives

$$W(s)^2 = W(r)^2 + \int_r^s \mathrm{d}t + \int_r^s 2W(t)\mathrm{d}W(t),$$

which implies the result.

ii) Observe that for a sequence of partitions $P = \{r = t_0 < t_1 < \dots < t_m = s\}$ such that $\max_{1\leqslant j\leqslant m} |t_j - t_{j-1}| \to 0$ we have

$$\int_r^s t\mathrm{d}W(t) = \lim_{m\to\infty} \sum_{j=1}^m t_{j-1}(W(t_j) - W(t_{j-1})) \quad \text{in } \mathrm{L}^2(\Omega).$$

As $t \mapsto W(t)$ is continuous a.s., we also have

$$\int_r^s W(t)\mathrm{d}t = \lim_{m\to\infty} \sum_{j=1}^m W(t_j)(t_j - t_{j-1}),$$

indeed since it is an ordinary Riemann sum, the integrand can be evaluated at any point in $[t_{j-1}, t_j]$. Therefore, we obtain

$$\begin{aligned}\int_r^s W(t)\mathrm{d}t + \int_r^s t\mathrm{d}W &= \lim_{m\to\infty} \sum_{j=1}^m t_{j-1}(W(t_j) - W(t_{j-1})) + W(t_j)(t_j - t_{j-1}) \\ &= rW(r) - sW(s),\end{aligned}$$

which shows the result.

Exercise 3.

Let F: $[a,b]\times\mathbb{R} \to \mathbb{R}$ be a continuous function with continuous partial derivatives $\partial F/\partial t, \partial F/\partial x, \partial^2 F/\partial x^2$ and let $(\partial F/\partial x)(t,x) = f(t,x)$. Show the following analogue of the fundamental theorem of the Leibniz–Newton calculus for the Itô calculus

$$\int_a^b f(t, W(t))\mathrm{d}W(t) = F(t, W(t))\Big|_a^b - \int_a^b \Big(\frac{\partial F}{\partial t}(t, W(t)) + \frac{1}{2}\frac{\partial f}{\partial x}(t, W(t))\Big)\mathrm{d}t.$$

In particular, when $F(t,x) = F(x)$ is independent of t, it reads

$$\int_a^b f(W(t))\mathrm{d}W(t) = F(W(t))\Big|_a^b - \frac{1}{2}\int_a^b f'(W(t))\mathrm{d}t.$$

Then, use this formula to compute

$$\int_0^t W(s)e^{W(s)}\mathrm{d}W(s).$$

Solution

The first result is obtained applying the Itô formula to compute $\mathrm{d}F(t, W(t))$. Then, let $f(x) = xe^x$, which gives

$$\begin{aligned}F(x) &= \int_0^x ye^y\mathrm{d}y = (x-1)e^x + 1, \\ f'(x) &= (x+1)e^x.\end{aligned}$$

The formula yields

$$\int_0^t W(s)e^W(s)\mathrm{d}W(s) = (W(t)-1)e^{W(t)} + 1 - \frac{1}{2}\int_0^t (1+W(s))e^{W(s)}\mathrm{d}s.$$

Exercise 4.

Compute $\mathbb{E}\Big[B_s \int_0^t B_u \mathrm{d}B_u\Big]$ and $\mathbb{E}\Big[B_s^2\Big(\int_s^t B_u \mathrm{d}B_u\Big)^2\Big]$ for $0 \leqslant s \leqslant t$.
Recall: If $\xi \sim \mathcal{N}(0,1)$, then $\mathbb{E}[\xi^4] = 3$.

Solution

Let $s \leqslant t$. We have $B_s = \int_0^t 1_{[0,s[}(v)dB_v$ therefore,

$$\mathrm{E}\bigg(B_s \int_0^t B_u dB_u\bigg) = \mathrm{E}\bigg(\int_0^t 1_{[0,s[}(v)dB_v \int_0^t B_u dB_u\bigg) = \int_0^t \mathrm{E}\big[1_{[0,s]}(u)B_u\big]du = 0$$

If $t \leqslant s$, the same argument leads to the same result. For the second moment, we have

$$\mathrm{E}\left[B_s^2\bigg(\int_s^t B_u dB_u\bigg)^2\right] = \mathrm{E}\left[\bigg(\int_s^t B_s B_u dB_u\bigg)^2\right] = \mathrm{E}\bigg[\int_s^t B_s^2 B_u^2 du\bigg]$$

Recalling that $s \leqslant u$ so that $B_u - B_s$ is independent of B_s,

$$\mathrm{E}(B_s^2 B_u^2) = \mathrm{E}\big[B_s^2(B_u - B_s + B_s)^2\big] = \underbrace{\mathrm{E}\big[B_s^2(B_u-B_s)^2\big]}_{=s(u-s)} + 2\underbrace{\mathrm{E}[B_s^3(B_u-B_s)]}_{=0} + \mathrm{E}(B_s^4).$$

We can write $B_s = \sqrt{s}Z$ where $Z \sim N(0,1)$ and therefore $\mathrm{E}(B_s^4) = s^2\mathrm{E}(Z^4) = 3s^2$. In conclusion, $\mathrm{E}(B_s^2 B_u^2) = s(u-s) + 3s^2$ and

$$\mathrm{E}\left[B_s^2\bigg(\int_s^t B_u dB_u\bigg)^2\right] = \int_s^t s(u-s) + 3s^2 du = \frac{1}{2}s(t-s)^2 + 3s^2(t-s).$$

Exercise 5.

Let $X \in M^2([0,T])$ and consider the stochastic integral $I_t = \int_0^t X_s dB_s$.

1) Use Itô's formula to show that

$$|I_t|^p = p\int_0^t |I_s|^{|p-1} \operatorname{sgn}(I_s) X_s dB_s + \frac{1}{2}p(p-1)\int_0^t |I_s|^{p-2} X_s^2 ds.$$

2) Assume $|I_t| \leqslant K$ a.s. for some $K > 0$. Deduce that there exists $c_p = C(T,p)$ such that

$$\mathrm{E}\bigg(\sup_{0\leqslant t\leqslant T}\bigg|\int_0^t X_s dB_s\bigg|^p\bigg) \leqslant c_p \mathrm{E}\left[\bigg(\int_0^T |X_s|^2 ds\bigg)^{\frac{p}{2}}\right] \leqslant c_p T^{\frac{p-2}{2}} \mathrm{E}\bigg(\int_0^T |X_s|^p ds\bigg)$$

for some constant $c_p > 0$.

Hint. Use Doob's martingale and Holder inequalities.

3) Show that 2) works without asking for $|I_t| \leqslant K$ a.s. .

Hint. Take $\tau_n = \inf\{t \leqslant T; |I_t| \geqslant n\}$ and consider the martingale $I_{t\wedge\tau_n}$. Then take the limit for $n \to \infty$ and conclude by Fatou's lemma.

Solution

1) Let us apply Ito's formula to the function $f(x) = |x|^p$ (which is twice differentiable, as $p \geqslant 2$) and to the process I whose stochastic differential is $dI_t = X_t dB_t$. We have $f'(x) = p\,\mathrm{sgn}(x)|x|^{p-1}, f''(x) = p(p-1)|x|^{p-2}$, where sgn denotes the "sign" function ($=1$ for $x \geqslant 0$ and -1 for $x < 0$). Then by Ito's formula

$$
\begin{aligned}
d|I_t|^p &= f'(I_t)dI_t + \frac{1}{2}f''(I_t)d\langle I\rangle_t \\
&= |I_s|^{|p-1}\,\mathrm{sgn}(I_s)X_s dB_s + \frac{1}{2}p(p-1)|I_s|^{p-2}X_s^2 ds,
\end{aligned}
$$

i.e., as $I_0 = 0$,

$$
|I_t|^p = p\int_0^t |I_s|^{|p-1}\,\mathrm{sgn}(I_s)X_s dB_s + \frac{1}{2}p(p-1)\int_0^t |I_s|^{p-2}X_s^2 ds.
$$

2) One can of course assume $X \in M^p([0,T])$, otherwise the statement is obvious (the right-hand side is $= +\infty$). Let $I_t = \int_0^t X_s dB_s$ and define $I_t^* = \sup_{0\leqslant s\leqslant t}|I_s|$. $(I_t)_t$ is a square integrable martingale and by Doob's inequality

$$
\mathrm{E}\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t X_s dB_s\right|^p\right) = \mathrm{E}[I_t^{*p}] \leqslant \left(\frac{p}{p-1}\right)^p \mathrm{E}[|I_T|^p].
$$

Let us now first assume $|I_T^*| \leqslant K$: this guarantees that $|I_.|^{p-1}\,\mathrm{sgn}(I_.)X_. \in M^2([0,T])$. Let us take the expectation recalling that the stochastic integral has zero mean. By Doob's inequality and Hölder's inequality with the exponents $\frac{p}{2}$ and $\frac{p}{p-2}$, we have

$$
\begin{aligned}
\mathrm{E}[I_T^{*p}] &\leqslant \left(\frac{p}{p-1}\right)^p \mathrm{E}[|I_T|^p] = \underbrace{\frac{1}{2}\left(\frac{p}{p-1}\right)^p p(p-1)}_{:=c_0}\mathrm{E}\left(\int_0^T |I_s|^{p-2}X_s^2 ds\right) \\
&\leqslant c_0\mathrm{E}\left(I_T^{*p-2}\int_0^T X_s^2 ds\right) \leqslant c_0\mathrm{E}[I_T^{*p}]^{1-\frac{2}{p}}\mathrm{E}\left[\left(\int_0^T X_s^2 ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}
\end{aligned}
$$

As we assume $|I_T^*| \leqslant K, \mathrm{E}[I_T^{*p}] < +\infty$ and in the previous inequality we can divide by $\mathrm{E}[I_T^{*p}]^{1-\frac{2}{p}}$, which gives

$$
\mathrm{E}[I_T^{*p}]^{\frac{2}{p}} \leqslant c_0\mathrm{E}\left[\left(\int_0^T X_s^2 ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}
$$

i.e.

$$
\mathrm{E}[I_T^{*p}] \leqslant c_0^{p/2}\mathrm{E}\left[\left(\int_0^T X_s^2 ds\right)^{\frac{p}{2}}\right] \leqslant c_0^{p/2}T^{\frac{p-2}{p}}\mathrm{E}\left[\left(\int_0^T |X_s|^p ds\right)\right].
$$

3) Let $\tau_n = \inf\{t \leqslant T; |I_t| \geqslant n\}$ ($\tau(n) = T$ if $\{\} = \varnothing).(\tau_n)_n$ is a sequence of stopping times increasing to T, as the paths of I are continuous and then also bounded. We have therefore $I_{\tau_n\wedge t} \to I_t$ as $n \to \infty$ and

$$
I_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} X_s dB_s = \int_0^t X_s 1_{\{s<\tau_n\}} dB_s,
$$

so that one has

$$\mathrm{E}\big(I^{*}_{T\wedge\tau_n}{}^{p}\big) \leqslant c_0^{p/2}\mathrm{E}\left[\left(\int_0^T |X_s|^2 1_{\{s<\tau_n\}} ds\right)^{\frac{p}{2}}\right] \leqslant c_0^{p/2}\mathrm{E}\left[\left(\int_0^T |X_s|^2 ds\right)^{\frac{p}{2}}\right]$$

and we can just apply Fatou's lemma. Finally, again by Hölder's inequality,

$$\mathrm{E}\left[\left(\int_0^T |X_s|^2 ds\right)^{\frac{p}{2}}\right] \leqslant T^{\frac{p-2}{p}}\,\mathrm{E}\left[\int_0^T |X_s|^p ds\right].$$

Exercise 6.

Consider the Itô process

$$X_t = X_0 + \int_0^t f_s\, ds + \int_0^t g_s\, dB_s = X_0 + J_t + I_t$$

where $\{B_t\}_t$ is a standard Brownian motion, $f \in M^1([0,T])$, and $g \in M^2([0,T])$.

Let us focus first on the process

$$I_t = \int_0^t g_s\, dB_s.$$

1) Show that I_t is a martingale.

2) Show that

$$\langle I\rangle_t = \int_0^t g_s^2\, ds,$$

where $\langle I\rangle_t$ is the quadratic variation of I_t (i.e. that $I_t^2 - \langle I\rangle_t$ is a martingale).

It can be shown that

$$\langle I\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(I_{t_{j+1}} - I_{t_j})^2 \text{ in probability,}$$

where $\pi = \{0 = t_0 < \cdots < t_n = t\}$ is any partition of $[0,t]$ and $|\pi| = \max_j |t_{j+1} - t_j|$.

We focus now on the process X_t. By definition, $\langle X\rangle_t = \langle I\rangle_t = \int_0^t g_s^2\, ds$.

3) Show that

$$\langle X\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})^2 \text{ in probability}$$

.

4) Consider another Itô process

$$Y_t = Y_0 + \int_0^t \tilde{f}_s\, ds + \int_0^t \tilde{g}_s\, dB_s = X_0 + \tilde{J}_t + \tilde{I}_t$$

with $\tilde{f} \in M^1([0,T])$, and $\tilde{g} \in M^2([0,T])$, by definition

$$\langle X, Y\rangle = \langle I, \tilde{I}\rangle_t = \int_0^t g_s \tilde{g}_s ds.$$

5) Show that

$$\langle X, Y\rangle = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j}) \text{ in probability,}$$

using that

$$\langle I, \tilde{I}\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(I_{t_{j+1}} - I_{t_j})(\tilde{I}_{t_{j+1}} - \tilde{I}_{t_j}) \text{ in probability.}$$

Solution

1) If $t > s$ one has

$$\mathrm{E}[I(t) - I(s) \mid \mathcal{F}_s] = \mathrm{E}\left[\underbrace{\int_s^t X_u dB_u}_{\mathcal{F}_s-\text{independent}} \mid \mathcal{F}_s\right] = 0 \quad \text{a.s.}$$

and therefore we have the martingale relation

$$\mathrm{E}[I(t) \mid \mathcal{F}_s] = I(s) + \mathrm{E}[I(t) - I(s) \mid \mathcal{F}_s] = I(s) \quad \text{a.s. .}$$

2) For the quadratic variation property we need to show that $I(t)^2 - \langle I\rangle_t$ is a martingale

$$\begin{aligned}
\mathrm{E}\big[I(t)^2 - \langle I\rangle_t \mid \mathcal{F}_s\big] &= \mathrm{E}\big[(I(t) - I(s))^2 + 2((I(t) - I(s))I(s)) + I(t)^2 - (\langle I\rangle_t - \langle I\rangle_s) - \langle I\rangle_s \mid \mathcal{F}_s\big] \\
&= \mathrm{E}\big[(I(t) - I(s))^2 \mid \mathcal{F}_s\big] + 2\mathrm{E}[(I(t) - I(s))I(s) \mid \mathcal{F}_s] + I(s)^2 - \mathbb{E}[\int_s^t X_s^2 ds] - \langle I\rangle_s \\
&= \mathbb{E}[(\int_s^t X_s dB_s)^2] + I(s)^2 - \mathbb{E}[\int_s^t X_s^2 ds] - \langle I\rangle_s \\
&= I(s)^2 - \langle I\rangle_s.
\end{aligned} \tag{6.1}$$

3) One has

$$\begin{aligned}
\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})^2 &= \sum_{j=0}^{n-1}(\int_{t_j}^{t_{j+1}} f_s\, ds + \int_{t_j}^{t_{j+1}} g_s\, dB_s)^2 \\
&= \underbrace{\sum_{j=0}^{n-1}(\int_{t_j}^{t_{j+1}} f_s\, ds)^2}_{T_1} + \underbrace{2\sum_{j=0}^{n-1}(\int_{t_j}^{t_{j+1}} f_s\, ds)(\int_{t_j}^{t_{j+1}} g_s\, dB_s)}_{T_2} + \underbrace{\sum_{j=0}^{n-1}(\int_{t_j}^{t_{j+1}} g_s\, dB_s)^2}_{T_3}
\end{aligned} \tag{6.2}$$

Since the function $t \to f_t(\cdot)$ is integrable in t ($f \in \mathcal{M}^1([0, T])$), let $N \in F$ be s.t. $\forall\omega \in N^C$ we have $N = \int_0^T f_s ds < \infty$ then $\forall\omega \in N^C$

$$\begin{aligned}
|T_1(\omega)| &\leqslant \max_j |I_{t_{j+1}} - I_{t_j}| \sum_{j=0}^{n-1} |\int_{t_j}^{t_{j+1}} f_s\, ds| \\
&\leqslant \max_j |I_{t_{j+1}} - I_{t_j}| |\int_0^t f_s\, ds| \to 0
\end{aligned} \tag{6.3}$$

Hence $T_1 \to 0$ a.s. and in probability. Similarly for T_2

$$\begin{aligned}
|T_2(\omega)| &\leqslant 2\max_j |I_{t_{j+1}} - I_{t_j}| |\int_{t_j}^{t_{j+1}} f_s\, ds| \\
&\leqslant 2\max_j |I_{t_{j+1}} - I_{t_j}| |\int_0^t f_s\, ds| \to 0
\end{aligned} \tag{6.4}$$

since I_t is continuous. Hence $T_2 \to 0$ a.s. and in probability. Finally, $T_3 \to \int_0^t g_s^2\, ds$ in probability by L^2 convergence of the construction of the stochastic integral.

4) Analogous to Point 3).