Lecture 8 - Asymptotic stability

We consider the d-dimensional SDE
$$dX_t = b(X_t,t)\,dt + \sigma(X_t,t)\,dB_t \qquad t>t_0, \quad X_{t_0} = x_0 \in \mathbb{R}^d$$
and assume that b, σ satisfy any of the assumptions (global Lipschitz, local Lipschitz + linear growth bound or monotonicity condition) that guarantee existence and uniqueness of a solution for all times $t \geq t_0$, with all moments bounded.
To highlight the dependence on the initial condition we will use sometimes the notation $X_t^{t_0,x_0}$. We further assume that
$$b(0,t)=0, \qquad \sigma(0,t)=0 \qquad \forall t \geq t_0$$
Hence $X_t^{t_0,0}=0$ is a solution (the trivial solution).
We want to study the stability of the trivial solution.
In the stochastic setting, several notions of stability can be introduced. We focus, in particular, on the notion of asymptotic stability by which we expect solutions starting from x_0 close to 0 to converge "in some sense" to 0 as $t \to \infty$.

<u>Definition</u>:

i) the trivial solution is said to be stochastically asymptotically stable if for any $\varepsilon \in (0,1)$, there exists $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ s.t.
$$\mathbb{P}\left(\lim_{t\to\infty} X_t^{t_0,x_0} = 0\right) \geq 1-\varepsilon$$
whenever $|x_0| < \delta_0$.

ii) the trivial solution is said to be stochastically asymptotically stable in the large if for any $x_0 \in \mathbb{R}^d$
$$\lim_{t\to\infty} X_t^{t_0,x_0} = 0 \qquad \text{a.s.}$$
It is almost surely exponentially stable if moreover
$$\limsup_{t\to\infty} \frac{1}{t}\log|X_t^{t_0,x_0}| < 0 \qquad \text{a.s.}$$

iii) the trivial solution is said to be mean-square stable if
$$\lim_{t\to\infty} \mathbb{E}[\|X_t^{t_0,x_0}\|^2] = 0$$

and exponentially mean-square stable if
$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}[\|X_t^{t_0,x_0}\|^2] < 0$$
A similar definition holds for pth moment stability with $p\neq 2$.

Let us consider a model problem of a linear one-dimensional SDE driven by an m-dimensional Brownian motion
$$dX_t = \mu X_t\, dt + \sum_{i=1}^m \sigma_i X_t\, dB_t^{(i)} \qquad t>t_0 \qquad X_{t_0}=x_0 \qquad (1)$$
with $\mu, \sigma_1, \dots, \sigma_m \in \mathbb{R}$

the solution is given by
$$X_t^{t_0,x_0} = x_0 \exp\left\{ \left(\mu - \frac{1}{2}\sum_{i=1}^m \sigma_i^2\right)(t-t_0) + \sum_{i=1}^m \sigma_i \left(B_t^{(i)} - B_{t_0}^{(i)}\right)\right\}$$
$$= x_0 \exp\left\{ \left(\mu - \frac{1}{2}\sum_{i=1}^m \sigma_i^2\right)(t-t_0)\right\} \prod_{i=1}^m e^{\sigma_i\left(B_t^{(i)} - B_{t_0}^{(i)}\right)}$$

taking the logarithm
$$\log|X_t^{t_0,x_0}| = \log|x_0| + \left(\mu - \frac{1}{2}\Sigma_i\sigma_i^2\right)(t-t_0) + \sum_{i=1}^m \sigma_i\left(B_t^{(i)} - B_{t_0}^{(i)}\right)$$

Recalling the law of iterated logarithms
$$\limsup_{t\to\infty} \frac{B_t^{(i)}}{\sqrt{2t\log\log t}} = 1, \qquad \liminf_{t\to\infty} \frac{B_t^{(i)}}{\sqrt{2t\log\log t}} = -1 \qquad \text{a.s.} \quad \forall i$$
we see that $\displaystyle\lim_{t\to\infty} \frac{1}{t}\left(B_t^{(i)} - B_{t_0}\right) = 0$ a.s. $i = 1, \dots, m$, hence
$$\lim_{t\to\infty} \frac{1}{t}\log|X_t^{t_0,x_0}| = \mu - \frac{1}{2}\Sigma_i\sigma_i^2 \qquad \text{a.s.}$$
We then conclude that (1) is stochastically asymptotically stable in the large (actually exponentially stable) if
$$\mu - \frac{1}{2}\Sigma_i\sigma_i^2 < 0$$

We can also easily analyze the mean square stability of (1). We have

$$\mathbb{E}[|X_t^{t_0,x_0}|^2] = |x_0|^2 \exp\{2(\mu - \tfrac{1}{2}\Sigma_i\sigma_i^2)(t-t_0)\} \prod_{i=1}^m \mathbb{E}[e^{2\sigma_i(B_t^{(i)} - B_{t_0}^{(i)})}]$$

$$= |x_0|^2 \exp\{2(\mu - \tfrac{1}{2}\Sigma_i\sigma_i^2)(t-t_0)\} \prod_{i=1}^m e^{2\sigma_i^2(t-t_0)}$$

$$= |x_0|^2 \exp\{2(\mu + \tfrac{1}{2}\Sigma_i\sigma_i^2)(t-t_0)\}$$

from which we conclude that (1) is (exponentially) mean square stable if $\mu + \frac{1}{2}\Sigma_i\sigma_i^2 < 0$.

Summarizing

$X_t = 0$ is (exponentially) asympt. stable $\iff \mu - \frac{1}{2}\Sigma_i\sigma_i^2 < 0$

$X_t = 0$ is (exponentially) mean square stable $\iff \mu + \frac{1}{2}\Sigma_i\sigma_i^2 < 0$

Notice that the condition for mean square stability is stronger than that for asymptotic stability. This is often the case, although the implication is not always true.

For more complex SDEs for which we don't have an explicit expression of the solution, a convenient way to analyze the stability of the trivial solution is by means of a Lyapunov function. We recall some sufficient conditions for stability involving a Lyapunov function:

<u>Theorem 1</u>: Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0,\infty); \mathbb{R}_+)$ and constants $p>0$, $c_1>0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$ s.t. $\forall x \neq 0$, $t \geq t_0$

- $c_1|x|^p \leq V(x,t)$
- $(\mathcal{L}_t V)(x,t) \leq c_2 V(x,t)$
- $|b^T(x,t)\nabla_x V(x,t)| \geq c_3 V(x,t)$

then $\limsup_{t\to\infty} \frac{1}{t}\log|X_t^{t_0,x_0}| \leq -\frac{c_3^2 - 2c_2}{2p}$ a.s.

Hence $X_t = 0$ is exponentially stable if $c_3^2 > 2c_2$

In the previous theorem, $\mathcal{L}_t$ is the generator associated to the SDE

$$(\mathcal{L}_t V)(x,t) = b^T(x,t)\nabla_x V(x,t) + \frac{1}{2}\sigma(x,t)\sigma^T(x,t) : \nabla^2_{xx} V(x,t)$$

<u>Theorem 2</u> Assume that there is a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0,\infty); \mathbb{R}_+)$ and constants $c_1, \hat{c}_1, c_2 > 0$ s.t. $\forall x \neq 0$, $\forall t \geq t_0$

- $c_1 |x|^p \leq V(x,t) \leq \hat{c}_1 |x|^p$
- $(\mathcal{L}_t V)(x,t) \leq -c_2 V(x,t)$

then $\mathbb{E}[\|X_t^{t_0,x_0}\|^p] \leq \frac{\hat{c}_1}{c_1}\|x_0\|^p e^{-c_2(t-t_0)}$

hence $X_t = 0$ is pth-moment exponentially stable (for $c_2 > 0$).

We mention also that under the monotonicity-type assumption on b, σ: $x^T b(x,t) + \|\sigma(x,t)\|^2 \leq k|x|^2 \quad \forall x \in \mathbb{R}^d, t \geq t_0$ for some $k > 0$, then exponential mean square (pth-moment) stability implies exponential asymptotic stability.

<u>Mean-square and asymptotic stability of numerical schemes</u>

We now consider a numerical approximation of the SDE

$$dX_t = b(X_t,t)\,dt + \sigma(X_t,t)\,dB_t, \quad t \geq t_0, \quad X_{t_0} = x_0$$

by a suitable numerical scheme that produces the sequence $\{Y_n\}_{n=0}^{\infty}$ where $Y_n \approx X_{t_n}$, $t_n = t_0 + n\Delta t$.

Assuming that the trivial solution is mean-squares / asymptotically stable, we may ask the question whether the numerical solution preserves this stability property, i.e. whether $\lim_{n\to\infty} \mathbb{E}[\|Y_n\|^2] = 0$ (mean-square stability)

or $\lim_{n\to\infty} |Y_n| = 0$ a.s. (asymptotic stability)

We restrict the study to the model problem analyzed earlier of a scalar one dimensional SDE with $m=1$

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t$$

for which we know that

$X_t = 0$ is mean-square stable iff $\mu + \frac{1}{2}\sigma^2 < 0$

asymptotically stable iff $\mu - \frac{1}{2}\sigma^2 < 0$

and focus on the stochastic θ-method

$$Y_{n+1} = Y_n + \left[\theta \mu Y_{n+1} + (1-\theta)\mu Y_n\right]\Delta t + \sigma Y_n \Delta B_n \qquad n = 0, 1, \dots$$

assuming $1 - \theta\mu\Delta t \neq 0$

Mean-square stability

We have $$Y_{n+1} = \frac{1}{1-\theta\mu\Delta t}\left[1 + (1-\theta)\mu\Delta t + \sigma\Delta B_n\right] Y_n$$

hence $$\mathbb{E}[Y_{n+1}^2] = \frac{1}{(1-\theta\mu\Delta t)^2}\,\mathbb{E}\left[\left(1+(1-\theta)\mu\Delta t + \sigma\Delta B_n\right)^2\right]\mathbb{E}[Y_n^2]$$

$$= \frac{\left(1+(1-\theta)\mu\Delta t\right)^2 + \sigma^2\Delta t}{(1-\theta\mu\Delta t)^2}\,\mathbb{E}[Y_n^2]$$

The scheme is therefore mean-square stable iff

$$\frac{\left(1+(1-\theta)\mu\Delta t\right)^2 + \sigma^2\Delta t}{(1-\theta\mu\Delta t)^2} < 1$$

$$\iff \left(1+(1-\theta)\mu\Delta t\right)^2 - (1-\theta\mu\Delta t)^2 < -\sigma^2\Delta t$$

$$\iff \mu\Delta t\left(2 + (1-2\theta)\mu\Delta t\right) < -\sigma^2\Delta t$$

$$\iff (1-2\theta)\mu^2\Delta t < -2\mu - \sigma^2 = -2\left(\mu + \tfrac{1}{2}\sigma^2\right)$$

We analyze separately the 3 cases $\theta \in [0, 1/2)$, $\theta = 1/2$, $\theta \in (1/2, 1]$

- $\theta \in [0, 1/2) \Rightarrow \Delta t < -\dfrac{2(\mu + 1/2\,\sigma^2)}{(1-2\theta)\mu^2}$

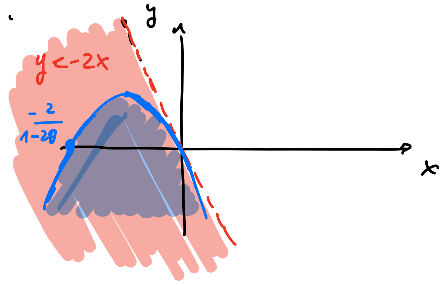
hence, if the SDE is MS stable, $\lambda = \mu + 1/2\,\sigma^2 < 0$ then the θ-method is conditionally MS stable for $\Delta t < \dfrac{2|\lambda|}{(1-2\theta)\mu^2}$.

On the other hand, if the SDE is MS unstable, the θ-method is also MS unstable for any Δt.

let $x = \Delta t \mu$, $y = \Delta t \sigma^2$

SDE stability region: $y < -2x$

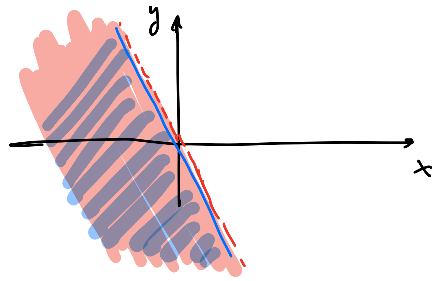
θ-method stab. region: $y < -(1-2\theta)x^2 - 2x$



- $\theta = 1/2$: SDE MS stable $\Rightarrow$ θ-method MS stable $\forall\, \Delta t > 0$
  SDE MS unstable $\Rightarrow$ θ-method MS unstable $\forall\, \Delta t > 0$

SDE stab. region $y < -2x$

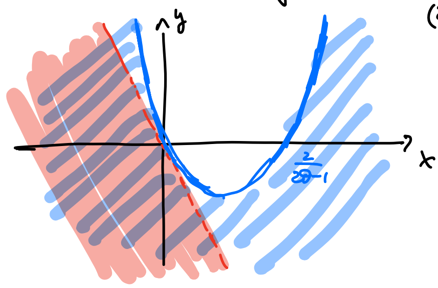
θ method stab. region $y < -2x$



- $\theta \in (1/2, 1]$ SDE MS stable $\Rightarrow$ θ method MS stable $\forall\, \Delta t > 0$
  SDE MS unstable $\Rightarrow$ θ method stable for $\Delta t > \dfrac{2(\mu + \frac{1}{2}\sigma^2)}{(2\theta - 1)\mu^2}$

SDE stab. region $y < -2x$

θ method stab. region $y < (2\theta-1)x^2 - 2x$



Asymptotic stability

We recall the scheme $Y_{n+1} = \dfrac{1 + (1-\theta)\mu \Delta t + \sigma \Delta B_n}{1 - \theta \mu \Delta t} Y_n = V_n Y_n$

with $V_n \sim N\left(\dfrac{1 + (1-\theta)\mu \Delta t}{1 - \theta \mu \Delta t}, \dfrac{\sigma^2 \Delta t}{(1 - \theta\mu \Delta t)^2} \right)$

Hence $Y_n = \left(\prod_{i=0}^{n-1} V_i \right) x_0$. Taking logarithms

$$\log |Y_n| = \log |x_0| + \sum_{i=0}^{n-1} \log |V_i|$$

Let $z_i = \log|V_i|$. Since z_i are iid for $i = 0, \dots, n-1$ and $E[|z_i|] < \infty$, by the strong law of large numbers

$\frac{1}{n}\sum_i z_i \xrightarrow{a.s.} E[z_1]$ and

$$\lim_{n\to\infty} \frac{1}{n}\log|Y_n| = \lim_{n\to\infty} \frac{1}{n}\log|x_0| + \frac{1}{n}\sum_{i=0}^{n-1}\log|V_i| = E[z_1] \quad a.s.$$

Hence • if $E[z_1] = E\left[\log\left|\frac{1+(1-\theta)\mu\Delta t + \sigma\Delta B}{1-\theta\mu\Delta t}\right|\right] < 0$ then $Y_n \xrightarrow{a.s.} 0$

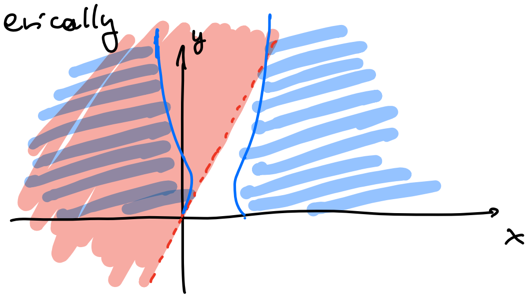
and the θ-method is asymptotically stable

• if $E[z_1] > 0$ then $\lim|Y_n| = \infty$ and the θ-method is asymptotically unstable.

The stability region $S = \{(x,y) : E\left[\log\left|\frac{1+(1-\theta)x+\sqrt{y}\,\xi}{1-\theta x}\right|\right] < 0, \ \xi \sim N(0,1)\}$

can be computed numerically

For $\theta = 1$



For $\theta = 0$