Exam

Duration : 180 minutes (15h15 – 18h15)

Last name :

First name :

SCIPER :

No documents allowed.
All electronic devices are forbidden.
Please do not remove the staple.
Ask for supplementary sheets of paper if needed.
Implement your code using MATLAB or Python on the desktop and **submit** your code for evaluation through Moodle. Only the files that will be submitted will be evaluated.

REMARK: All your answers should be justified in a **clear** and **synthetic** way.

Problem 1

Let $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $(W(t))_{t\geq 0}$ be a one-dimensional standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

i) Let $X_1(t) = \dfrac{1}{a}W(a^2 t)$ with $a \neq 0$. Show that $(X_1(t))_{t\geq 0}$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_{a^2 t}\}_{t\geq 0}$.

ii) Let $X_2(t) = \begin{cases} tW\left(\dfrac{1}{t}\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$ Show that $(X_2(t))_{t\geq 0}$ is a standard Brownian motion with respect to its own filtration.

(*Hint*: it is enough to check that X_t has the law of a Brownian motion).

iii) Let $X_3(t) = \displaystyle\int_0^t W(s)\,\mathrm{d}W(s)$. Compute $\mathbb{E}[X_3(t)]$ and $\mathbb{V}\text{ar}[X_3(t)]$.

iv) Check whether the following processes $(X(t))_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$ martingales

(a) $X(t) = W(t) + 4t$

(b) $X(t) = W(t)^2 - t$

(c) $X(t) = W(t)^3$

(d) $X(t) = t^2 W(t) - 2\displaystyle\int_0^t sW(s)\,\mathrm{d}s$.

Problem 2

Consider the following stochastic differential equation

$$\begin{cases} dX_t &= \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right) dt + \sqrt{1+X_t^2} dW_t., \\ X(0) &= X_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion.

i) Comment if (1) has solutions.

ii) Derive the stochastic differential of $Y_t = \log\left(\sqrt{1+X_t^2} + X_t\right)$ and, hence, derive a solution of (1).

iii) Given $T > 0$, estimate an upper bound to the following quantity:

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^2].$$

Problem 3

Consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) & = f(t, X(t))\mathrm{d}t + g(t, X(t))dW_t \\ X(t_0) & = 2, \end{cases} \tag{2}$$

where $\{W_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion. The stochastic Heun-method applied to (2) is defined as

$$\begin{aligned} Y_{n+1} = Y_n &+ \frac{1}{2}\left[f(t_n, Y_n) + f(t_{n+1}, Y_n + \Delta t f(t_n, Y_n) + g(t_n, Y_n)\Delta W_n)\right]\Delta t \\ &+ \frac{1}{2}\left[g(t_n, Y_n) + g(t_{n+1}, Y_n + \Delta t f(t_n, Y_n) + g(t_n, Y_n)\Delta W_n)\right]\Delta W_n \end{aligned} \tag{3}$$

Assume that f, g are globally Lipschitz, satisfy the linear-growth bound and are smooth.

i) Estimate the order of the local truncation error

$$\mathbb{E}[|Y_{n+1} - \tilde{X}(t_{n+1})|^2],$$

where Y_n is the approximation obtained by (3) at time $t_n = t_0 + n\Delta t$ and $\tilde{X}(t)$ is the the exact solution of (2) starting at the point $X(t_n) = Y_n$ (*Hint*: use Taylor expansion on (3) and the Itô-Taylor expansion of the exact solution $\tilde{X}(t_{n+1})$.)

ii) Let $T = 1$. Implement the method (3) for the following SDE

$$\begin{cases} \mathrm{d}X_t & = (-X(t) + 2t)\,\mathrm{d}t + dW_t \\ X(t_0) & = 2, \end{cases}$$

for $\Delta t = 0.1, 0.05, 0.01, 0.005, 0.001$ and $M = 200$ realizations and plot the error $\mathbb{E}[|Y_N - X(T)|^2]$ versus Δt. Which is the order of convergence that you observe? Comment the results.

iii) Can you find more restrictive assumptions on f and/or g in (2) under which the local truncation error is of higher order than what derived in *i*)?

Problem 4

Let $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space and $(W(t))_{t\geq 0}$ be a one-dimensional standard $\{\mathcal{F}_t\}$-Brownian motion. Consider the following SDE

$$\begin{cases} \mathrm{d}X(t) & = k(\alpha - \log(X(t)))X(t)\mathrm{d}t + \sigma X(t) dW_t, \ t \in [0,1] \\ X(0) & = X_0 > 0, \end{cases} \tag{4}$$

where $k, \alpha, \sigma \in \mathbb{R}$.

i) Using the change of variable $Y(t) = \log(X(t))$, compute the exact solution of (4) as well as $\mathbb{E}[X(t)]$. (*Hint*: recall that the expected value $\mathbb{E}[Z]$ of a lognormal random variable $Z = e^W$, $W \sim \mathcal{N}(\mu, \sigma^2)$ is $\mathbb{E}[Z] = \mu - \frac{1}{2}\sigma^2$)

ii) Let $X(0) = 1$, $k = 0.4$, $\sigma = 0.5$, $\gamma = 0.1$, $\alpha = 1$. Implement the Euler-Maruyama method for $\Delta t = 0.1, 0.05, 0.01, 0.005, 0.001$ and $M = 2000$ realizations and estimate the order of convergence of the weak error $|\mathbb{E}[X(t)] - \mathbb{E}[Y_N]| \approx \left|\mathbb{E}[X(t)] - \hat{\mathbb{E}}[Y_N]\right|$, where $\hat{\mathbb{E}}$ is the Monte Carlo estimator and $\{Y_n\}_n$ is the Euler-Maruyama solution, by plotting it in a log-log scale.

iii) Discuss how the sample size M in the computation of $\hat{\mathbb{E}}[Y_N]$ should be chosen as a function of Δt to make the Monte Carlo error comparable to the weak error.

iv) Repeat point *ii*) with the choice of $M = M(\Delta t)$ proposed in point *iii*). Comment the results.

Problem 5

Consider the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t))\,\mathrm{d}t + g(t, X(t))\,\mathrm{d}W(t), \quad 0 \le t \le T, \\ X(0) &= X_0, \end{aligned}$$

and the numerical approximation $\left\{Y_n^{\Delta t}\right\}_{n\ge 0}^{N}$ given by the Euler-Maruyama method with time-step Δt, where $Y_n^{\Delta t}$ approximates $X(n\Delta t)$, $n = 0, \dots, N = \frac{T}{\Delta t}$. Suppose that we want to approximate the quantity $Z = \mathbb{E}\left[\varphi\left(X(T)\right)\right]$, where φ is some sufficiently smooth function. Let $\overline{Z}$ be the multilevel Monte Carlo estimator defined by

$$\overline{Z} = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \left(\varphi_\ell^{(i,\ell)} - \varphi_{\ell-1}^{(i,\ell)}\right),$$

with $\varphi_{-1}^{(i,\ell)} \equiv 0$, $\varphi_\ell^{(i,\ell)} = \varphi\left(Y_{N_\ell}^{\Delta t,(i)}\right)$, $\Delta t_\ell = T/N_\ell$, and where we use the same Brownian path for $\varphi_\ell^{(i,\ell)}$ and $\varphi_{\ell-1}^{(i,\ell)}$, but independent Brownian paths for $\varphi_\cdot^{(i,\ell)}$ and $\varphi_\cdot^{(j,k)}$, if $i \neq j$ or $\ell \neq k$.

i) Show that the mean square error $\mathrm{MSE}\left(\overline{Z}\right) := \mathbb{E}\left[(Z - \overline{Z})^2\right]$ can be decomposed into the sum of the variance and the square of the bias. Characterize further the variance as the sum of contributions over the levels $l = 0, \dots, L$.

ii) Show that for the bias we have

$$\left(\mathrm{bias}\left(\overline{Z}\right)\right)^2 \le C\Delta t_L^2,$$

where Δt_L is the stepsize of the finest level.

iii) Show that $\mathbb{V}\mathrm{ar}\left(\varphi_\ell - \varphi_{\ell-1}\right) \le C\Delta t_\ell$ for a constant $C > 0$. Use this to give a recipe to choose the number of samples per level M_ℓ in order to obtain

$$\mathbb{V}\mathrm{ar}\left(\overline{Z}\right) \le C\Delta t_L^2 \frac{L+1}{L}.$$

iv) Assuming that we want to achieve an accuracy of $\mathrm{MSE}\left(\overline{Z}\right) = \mathcal{O}\left(\varepsilon^2\right)$, derive the corresponding computational cost of the multilevel Monte Carlo estimator with the choice of M_ℓ proposed in *iii)*.

v) Explain briefly, in what sense the multilevel Monte Carlo is a better estimator than the standard Monte Carlo one. No additional computations are necessary.

vi) Suppose that we want to apply the MLMC approach based on the Euler-Maruyama method to stiff problems. What issue might we encounter? Give a modified version of the MLMC estimator, still using EM, for such a scenario.

vii) What kind of numerical scheme could we consider instead of the Euler-Maruyama method, so that we can still use the estimator $\overline{Z}$ for stiff problems? Explain what additional computational cost arises with the new numerical scheme.

Additional paper